Quick! Find a Solution to the Brachistochrone Problem

MATHEMATICAL CURIOSITIES
By Mark Levi

The brachistochrone problem asks to find the "curve of quickest descent," and so it would be particularly fitting to have the quickest possible solution. The problem is to find the shape of the perfectly slippery trough between two points A and B such that a bead released at A will reach B in the least time in a uniform gravitational field (Figure 1). The following solution (stating that the answer is a cycloid) may not be the quickest there is, but it is the quickest one I know.

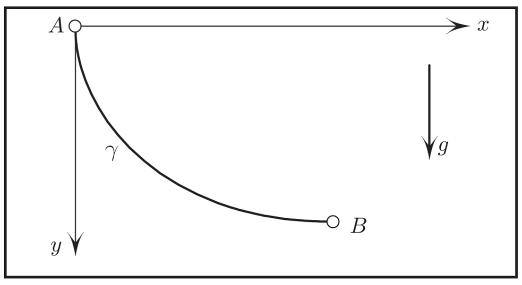

Figure 1.

Figure 2 shows the cycloid swept out by a point P on the rim of a circular wheel rolling on the ceiling. Let PC' be the tangent at P, with C' lying on the circle. Note that $PC' \perp CP$: The velocity of a point on the rigid body is perpendicular to the point's radius vector relative to the instantaneous center of rotation C. We conclude that CC' is a diameter. But in that case,

$$y = CP\sin\theta = D\sin^2\theta, \quad (1)$$

where θ is the angle between the tangent and the vertical.

Returning to the bead, its sliding time along γ is $\int ds/v$. Given that $v = \sqrt{2gy}$ (using conservation of energy and the assumption $v_A = 0$), this time is

$$k\int_\gamma \frac{ds}{\sqrt{y}}, \quad (2)$$

where k is a constant we don't care about.

Now, (2) is of the form $\int F(y)\,ds$, and minimizers of such functionals satisfy $F(y)\sin\theta$ = constant (a short calculus-free derivation of this can be found in [1]); for (2) this amounts to

$$\frac{\sin\theta}{\sqrt{y}} = \text{constant}, \quad (3)$$

which is the same equation as (1)!

The cycloid is thus a critical curve for the time functional (2) (although this does not prove minimality; a proof can be found in almost any book on calculus of variations, e.g., in [1]).

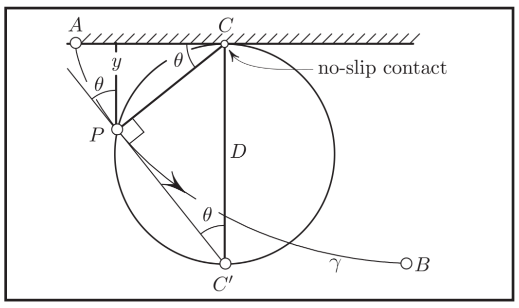

Figure 2.

References

[1] M. Levi, *Classical Mechanics with Calculus of Variations and Optimal Control*, AMS, Providence, Rhode Island, 2014.

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