Measuring Areas with a Shopping Cart

MATHEMATICAL CURIOSITIES

By Mark Levi

One late night in a deserted supermarket, I was waiting in a check-out aisle for a cashier. With no one around and nothing better to do, I began rolling the shopping cart's front wheel around the outline of a floor tile. The shopping cart ended up rotated after one traversal, as Figure 1 illustrates (for a round "tile" and with a "bike" instead of a shopping cart). It then dawned on me that the angle θ, by which the cart turns in one cycle (see Figure 1), is proportional to the area A of the tile, up to a small relative error if the diameter ε of the curve (not necessarily a circle) is small:

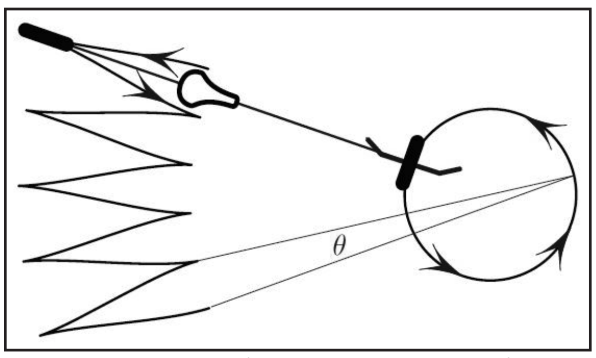

Figure 1. *Trajectories of the two wheels as the front one repeatedly circumscribes a closed curve.*

$$A = \theta L^2 + O(\epsilon^3), \qquad (1)$$

where L is the length of the "bike."

As happens with nearly every observation, someone noticed this before. Holger Prytz (a Danish cavalry officer) proposed the idea of calculating areas over 100 years earlier, and without the advantage of a shopping cart. One can find a beautiful description of this in [2], along with the discovery that the bike's direction angle changes after a cycle according to the Möbius transformation restricted to a circle,

$$z \mapsto e^{i\alpha}\frac{z-a}{1-\bar{a}z},$$

where α and a are functionals of the front path. The Prytz planimeter, also called the hatchet planimeter, is sketched in Figure 2. The hatchet moves like a rear wheel, not sliding sideways. The needle is guided around the region's boundary, and the angle θ gives the area according to (1).

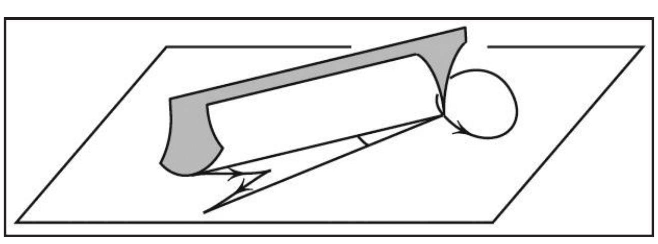

Figure 2. *The hatchet planimeter; (1) gives the area.*

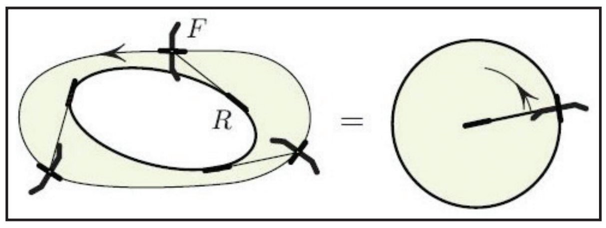

Figure 3. *RF is a directed segment with R's velocity constrained to the line RF.*

A geometrical explanation of (1) rests on two observations of independent interest.

Observation 1

The signed area[1] swept by a moving segment (as described in Figure 3) remains unchanged if the longitudinal velocity is altered (and in particular made to vanish).

This statement is intuitively plausible since the longitudinal motion has no effect on the rate at which RF sweeps the area.

[1] The area counts with the positive sign if the motion is to the left of RF, as it is in Figure 3.

Observation 2

A segment PQ executing a cyclic motion in the plane sweeps the signed area $A_Q - A_P$ (see Figure 4).

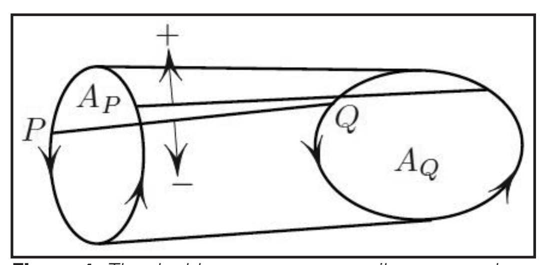

Figure 4. *The doubly-swept area contributes zero, leaving $A_Q - A_P$ as the net swept area.*

Proof of Prytz's Formula (1)

Figure 5 shows the motion of RF over one zig-zag, with F returning to its starting position; the rotation around the start/stop point through θ completes the cyclic motion, bringing RF to its initial position (the rotation violates the no-slip condition).

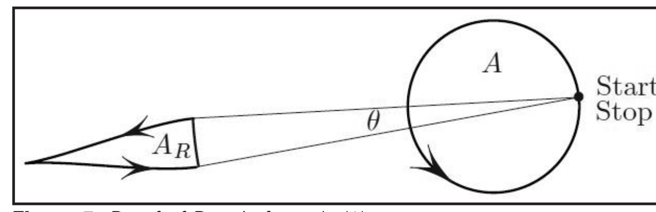

Figure 5. *Proof of Prytz's formula (1).*

During the "sliding" stage, RF sweeps area $\frac{1}{2}\theta L^2$, according to Observation 1. During the "rotating" stage, RF sweeps the sector of area $\frac{1}{2}\theta L^2$. The total swept area is thus θL^2. But this area equals $A - A_R$ by Observation 2, so that

$$A - A_R = \theta L^2.$$

It is not hard to show that $A_R = O(\varepsilon^3)$; this completes the outline of the proof of (1).

It is worth noting that if one extends the "bike" to $\mathbb{R}^3$, Prytz's formula (1) admits an eye-opening explanation entirely different from the one I just described. This explanation (given in [1]) is similar in spirit to the explanation of the finiteness of the radius of convergence of $1/(1+x^2)$ by extending to the complex domain.

As a concluding remark, choosing a circular annulus in Figure 3 yields another proof of the Pythagorean theorem (modulo the proof of Observation 1), as Figure 6 illustrates.

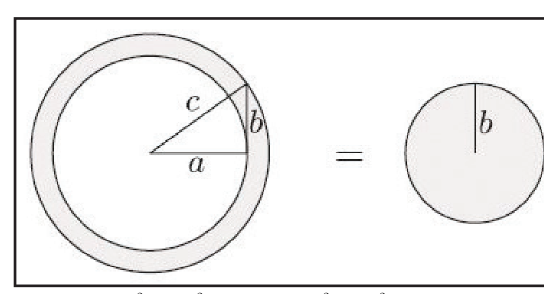

Figure 6. $\pi c^2 = \pi a^2 + \text{ring} = \pi a^2 + \pi b^2$.

The figures in this article were provided by the author.

References

[1] Bor, G., Levi, M., Perline, R., & Tabachnikov, S. (2017). Tire tracks and integrable curve evolution. Preprint, *arXiv:1705.06314*.

[2] Foote, R. (1998). Geometry of the Prytz planimeter. *Rep. Math. Phys., 42*, 249-271.

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