A Perspective on Altitudes

MATHEMATICAL CURIOSITIES

By Mark Levi

In my "geometry for teachers" class a few years ago, I was trying to explain why the altitudes in a triangle are concurrent. The (perhaps) most common proof, which identifies the concurrency point as the orthocenter of another larger triangle, still felt insufficiently direct to me. I was also wondering whether a more direct geometrical characterization of the concurrency point exists.

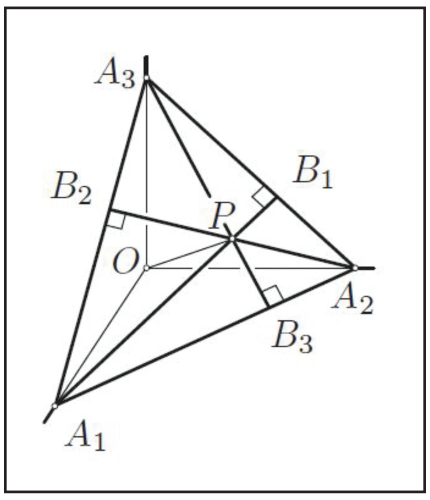

Figure 1. *Pushing the triangle into a corner. B_i is the point at which the line A_iP intersects with the opposite side.*

As it turns out, embedding the problem in three dimensions yields an additional insight. To begin, we shove an arbitrary acute[1] triangle $A_1A_2A_3$ into the corner of a rectangular quadrant, as shown in Figure 1; each vertex now lies on a coordinate axis.

I claim that the concurrency point of the altitudes is precisely the foot P of the perpendicular from the origin onto the plane of the triangle.

Proof

With P defined as in the previous sentence, let B_1 be the point at which the line A_1P intersects with side A_2A_3.

I claim that A_1B_1 is an altitude of the triangle. Indeed, $A_2A_3 \perp OP$ (since $A_2A_3 \in \text{plane}(A_1A_2A_3) \perp OP$) and $A_2A_3 \perp OA_1$ (since $A_2A_3 \in \text{plane}(OA_2A_3) \perp OA_1$). In summary, because A_2A_3 is normal to two lines (OP and OA_1) in the plane POA_1, it is normal to every line in that plane and thus to A_1B_1. So A_1B_1 is indeed an altitude. The same argument applies to A_iB_i for $i=2,3$, meaning that all altitudes pass through P. Q.E.D.

1 Unfortunately, this approach does not seem to extend to obtuse triangles; or perhaps I am not acute enough to find an extension.

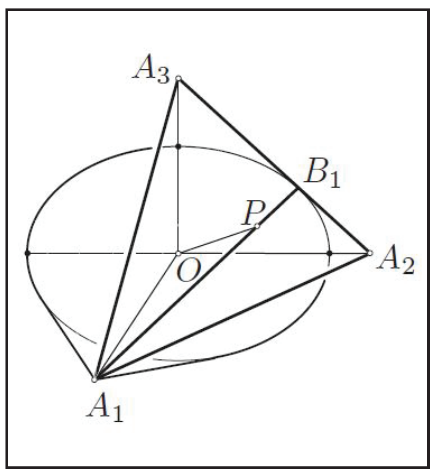

Figure 2. *The tangent cone.*

Proof 2

Here is a slightly different way to express essentially the same idea. Referring to Figure 2, construct the circular cone tangent to the plane of the triangle, with vertex A_1 and axis A_1O. Define B_1 as the point at which the line of tangency intersects with side A_2A_3. Now $A_1B_1 \perp A_2A_3$, according to Figure 3a, and A_1B_1 passes through P, according to Figure 3b (with P defined as above). This shows that the altitude from an arbitrarily chosen vertex passes through P. Q.E.D.

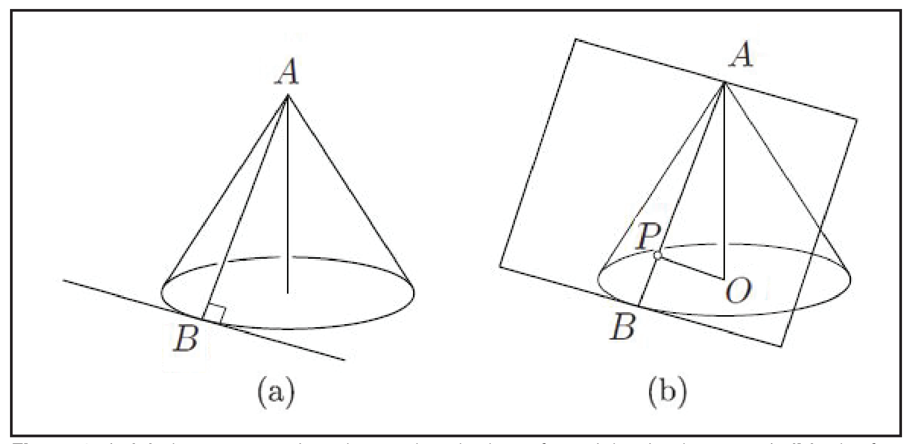

Figure 3. *In* **(a)***, the generator is orthogonal to the base for a right circular cone. In* **(b)***, the foot P of the perpendicular to a tangent plane from a point O on the axis lies on the line of tangency.*

The figures in this article were provided by the author.

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