siam news
Volume 48/ Number 8/ October 2015

Moser's theorem on the Jacobians

MATHEMATICAL CURIOSITIES
By Mark Levi

In one of his seminal papers [1], Moser proved a result, which in the simplest setting, still capturing the gist, states:

Given a positive continuous smooth function h on a compact, connected domain $D \subset R^n$ *with the average [h] = 1, there exists a diffeomorphism* φ *of D onto itself with the Jacobian h:*

$$\det \varphi'(x) = h(x) \quad (1)$$

Solving this nonlinear PDE for the components of φ may seem like a difficult problem, but a physical analogy leads to a solution at once, as follows.

Interpreting h as the initial density of a chemical dissolved in a medium occupying the domain D, we imagine that the chemical diffuses, equalizing its density as $t \to \infty$ *(the limiting density has to be 1 since [h] = 1). The map* φ *which sends each particle from t =0 to its position at* $t \to \infty$ *then satisfies (1).*

In a bit more detail, let the density $\rho = \rho(x, t)$ evolve according to the heat equation

$$\rho_t = \Delta\rho \quad (2)$$

with Neumann boundary conditions (no diffusion through ∂D), starting with $\rho(x, 0) = h(x)$. Assume that each particle $z = z(t)$ diffuses according to

$$\rho \dot{z} = -\nabla\rho; \quad (3)$$

such evolution preserves the mass $\int_{\Omega_t} \rho dV$ of any region Ω_t.

Thus $h\, dV_0 = \rho(x, t)\, dV_t$,

i.e. $$\frac{dV_t}{dV_0} = \frac{h}{\rho}.$$

In the limit $t \to \infty$ this turns into (1). The "diffusing particle" map φ solves the nonlinear PDE[1].

The missing details of this proof are not hard to fill in, or to find in [2].

There has been a lot of work on this problem since Moser's original paper, in particular on the regularity (references can be found in, e.g., [3]), but my modest goal here was to give a simple basic idea rather than a review of the latest results.

[1] Moser, J. *On the volume elements on a manifold*, Trans. Amer. Math. Soc. 120, 286-294 (1965).

[2] Levi, M. *On a problem by Arnold on periodic motions in magnetic fields*, Comm. Pure and Applied Mathematics. 56 (8), 1165–1177 (2003).

[3] Dacorogna, B and Moser, J. *On a partial differential equation involving the Jacobian determinant*. Ann. l'inst. H. Poincaré Anal. non linéaire. 7(1), 1-26 (1990).

1 indeed, the mass enters an infinitesimal patch dV at the rate $-\mathrm{div}(\mathrm{flux})dV \overset{(14)}{=} \Delta\rho dV$, precisely in agreement with (2). Formally, differentiating the mass integral gives two terms which cancel each other.

Food Security

continued from page 1

(3) How can we design a market system so that food prices embody the externalities (social and environmental costs) of food choices?

These questions were selected through a voting process from a longer list of 14 questions suggested by the participants on the first day of the workshop (see sidebar).

The (often lively) discussions yielded insight into the nature of the questions, relevant metrics, the availability of quantitative data (or lack thereof), and the modeling options. Clear cross-cutting themes emerged: interdisciplinary research, hybrid modeling, data mining, etc. Also, it soon became apparent that there is no hope for a one-size-fits all approach to the questions. For example, the discussions on the incorporation of externalities in food prices (question 3) brought to light the fact that products as diverse as corn and shrimp pose very different challenges: while the government has significant authority to regulate the price of a domestic staple product like corn, it has very little control over the externalities of shrimp, which is mostly produced abroad and imported.

Most interesting, workshop participants identified a long list of exploratory projects suitable for research with graduate and undergraduate students, including summarizing and visualizing data sets; constructing heat maps representing particular indices from nutrition databases; designing agent-based models to simulate behavior and choice processes; coupling conceptual dynamical-systems models, agent-based models, and Bayesian network models; and designing an object-oriented framework for modeling the food system.

The AIM workshop was the start of an effort to bring a new area of applications to the attention of the mathematics and computational science communities. More needs to be done. Food security will be one of the themes at the inaugural conference of the newly formed SIAM Activity Group on Mathematics of Planet Earth (SIAG/MPE), which will be held in Philadelphia, September 30–October 2, 2016.

References

[1] T. Acharya et al., *Assessing Sustainable Nutrition Security: The Role of Food Systems,* The International Life Sciences Institute, Research Foundation, Center for Integrated Modeling of Sustainable Agriculture and Nutrition, Washington, DC, June 2014; http://goo.gl/gEyQ1F.

Notes

[2] http://www.who.int/trade/glossary/story028/en/

[3] http://aimath.org/pastworkshops/foodsystem.html

Hans Kaper, founding chair of SIAG/MPE and editor-in-chief of SIAM News, *is an adjunct professor of mathematics at Georgetown University.*

14 Questions Related to Food Security

(1) What drives dietary inequality in the US?

(2) What strategies can we implement to create a more self-sustaining highly urbanized population with few immediate agricultural resources in the environment?

(3) How do we make market prices for food reflect the true cost?

(4) How do we maintain adequate water supply for all stakeholders?

(5) What can the US do to preemptively protect against disaster or attack of the US food distribution system?

(6) How can agriculture in California represent food supply nationally by growing suitable crops that are wanted and needed in the diet?

(7) If the US diet transitions to align with the USDA healthy eating guidelines, what policies would facilitate that transition without increasing environmental impacts?

(8) How would national regulation of greenhouse gas emissions impact food security in the US?

(9) What is the role of US policies and programs in contributing to a healthy food system that operates within planetary boundaries?

(10) How can we inform and empower consumers so they can make informed decisions with their dollars about the systemic impact of their dollars?

(11) How do we ensure equitable access to and availability of nutritious food for all in a growing population?

(12) How does one capture the ethical dimension of hunger in modeling the food system?

(13) What are the technologies and resources (use of land, water, etc.) needed to sustainably and resiliently maintain and improve food security (of food produced by US)?

(14) How will emerging economies (such as China) affect the US food system?

siam news

ISSN 1557–9573.