Gromov's Non-squeezing Theorem and Optics

MATHEMATICAL CURIOSITIES

By Mark Levi

A remarkable theorem discovered by Gromov ([1], [2]) states that it is impossible to map the unit ball $q_1^2+p_1^2+q_2^2+p_2^2 \le 1$ in $\mathbb{R}^4$ into a cylinder $q_1^2+p_1^2 \le r^2$ of smaller radius $r<1$ by a symplectic mapping.[1] This general statement has a surprising implication in optics.

Any optical device (a system of lenses and mirrors) gives rise to a symplectic map in $\mathbb{R}^4$ as follows. Placing the device between two parallel planes (Figure 1), we specify an incoming ray by the coordinates (q_1, q_2) of its crossing the first plane, and by the direction sines $(sin\,\theta_1, sin\,\theta_2)=(p_1, p_2)$. The ray is thus characterized by the point $x=(q_1,q_2,p_1,p_2)\in \mathbb{R}^4$. Let $X=(Q_1,Q_2,P_1,P_2)$ be the similarly-defined "exit data" of the same ray. Now it turns out that the mapping φ from x to X is symplectic, preserving the symplectic form ω mentioned in the footnote. (An intuitive proof, along with a physical interpretation of ω, can be found in [3]).

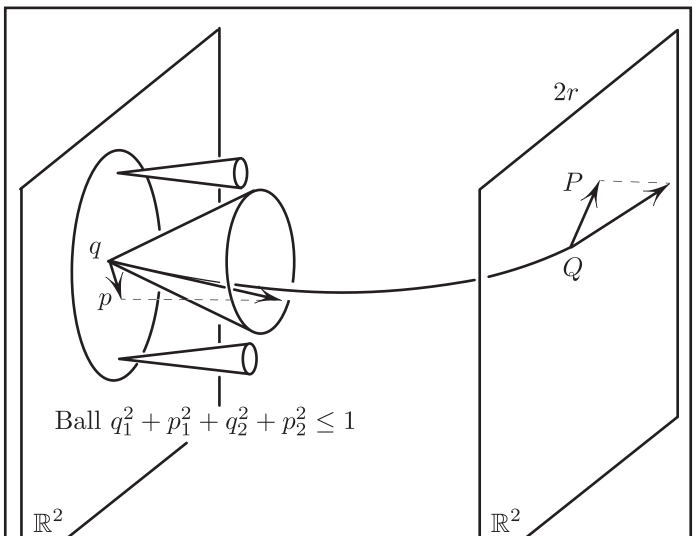

Figure 1.

With such parametrization of rays, the unit ball in $\mathbb{R}^4$ can be seen on the left side of Figure 1: at each point $q=(q_1,q_2)$ we have a cone of rays whose direction sines satisfy $p^2 \le 1-q^2$ (in particular, the cones become narrower near the edge of the unit disk). The cylinder $Q_1^2+P_1^2 \le r^2$ (Figure 2) corresponds to the set of rays exiting through the slit $|Q_1|\le r$ and confined to the dihedral angle with $|P_1|=|\sin\Theta_1|\le\sqrt{1-r^2}$. Again, the aperture of this angle decreases with the distance to the edge of the slit.

And thus Gromov's theorem implies the surprising fact that no optical device can shepherd the unit "ball" of incoming rays (Figure 1) through a narrow slit and with a narrow dihedral angle, as described more precisely in the preceding sentence.

Speaking of applications, the first application of Gromov's theorem to PDEs can be found in the remarkable paper [4].

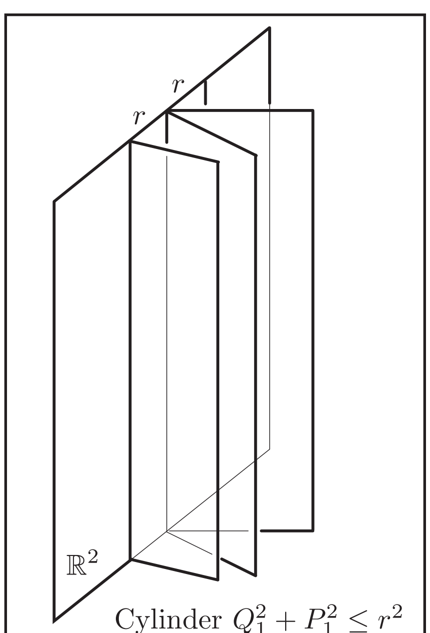

Figure 2.

[1] More precisely, by the map preserving the form $\omega = dq_1\wedge dp_1 + dq_2 \wedge dp_2$. Geometrically, this amounts to the requirement that for any infinitesimal parallelogram the sum of signed areas of its projections onto the planes (q_1,p_1) and (q_2,p_2) is preserved under the map.

References

[1] Gromov, M. (1985). Pseudo holomorphic curves in symplectic manifolds. *Inventiones Mathematicae, 82*, 307-347.

[2] Hofer, H., & Zehnder, E. (1994). *Symplectic Invariants and Halmiltonian Dynamics.* Birkhauser.

[3] Levi, M. (2014). Classical Mechanics with Calculus of Variations and Optimal Control: an Intuitive Introduction. *AMS.*

[4] Kuksin, S. (1995). Infinite Dimensional Symplectic Capacities and a Squeezing Theorem for Hamiltonian PDEs. *Commun. Math. Phys. 167*, 531-552.

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