Hanging Cables and Hydrostatics

MATHEMATICAL CURIOSITIES
By Mark Levi

Tension Is Linear

When driving down a country road and seeing hanging electric cables by the roadside, I marvel at nature's ability to solve a minimization problem; out of all possible shapes, it finds the one with the least potential energy. These hyperbolic cosine-shaped cables have another interesting property: their tension depends linearly on the height h (see Figure 1):

$$T - T_0 = \rho g h. \qquad (1)$$

Here, ρ is the cable's linear density — i.e., the mass per unit length. This is reminiscent of Pascal's law $p - p_0 = \rho g h$ for the water pressure at depth h; in this case—unlike in (1)—ρ stands for the water's density. It is not a coincidence; one can think of the hanging rope as a one-dimensional fluid wherein the tension corresponds to pressure in the water and unstretchability corresponds to incompressibility. In fact, the same energy-conservation argument proves both (1) and Pascal's law. The argument goes as follows. I begin by holding a stationary chain by its two ends, then advance each end by small distance ds in such a way that every particle of the chain advances along the curve (see Figure 2).

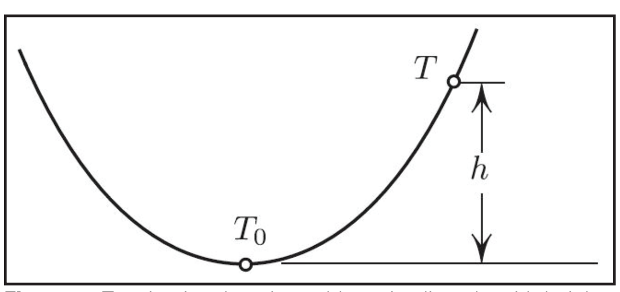

Figure 1. *Tension in a hanging cable varies linearly with height.*

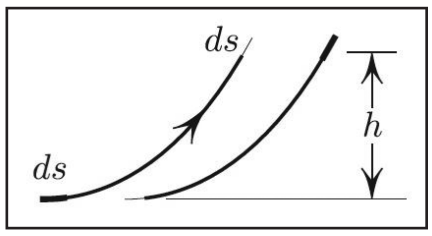

Figure 2. *Advancing the cable by ds changes its potential energy by $dm\,gh = \rho ds\,gh$ and takes work $(T - T_0)ds$.*

The advancing hand did work $T\,ds$ and the retreating hand did work $-T_0\,ds$; the minus is due to the fact that the applied force points against the displacement. The result of this motion is now the same as simply raising the element ds by height h, with the change of potential energy $dm\,gh = \rho ds\,gh$. Therefore,

$$T\,ds - T_0\,ds = \rho ds\,gh,$$

which amounts to (1). One can apply the exact same argument to prove Pascal's law, although textbooks do not usually take this approach.

Curvature and Tension

Can one "see" the tension T_0 at the lowest point of the cable? If one knows ρ (the linear density), the answer is yes. T_0 is the radius R of curvature, up to a factor:

$$T_0 = \rho g R. \qquad (2)$$

Indeed, the weight of the segment ds in Figure 3 is supported by the vertical tension:

$$T \sin d\theta = \rho g ds \quad \text{or} \quad T\frac{\sin d\theta}{ds} = \rho g.$$

With $ds \to 0$, this becomes $T_0 k = \rho g$ — where k is the curvature. This amounts to (2).

By combining (1) and (2), we get $T = m\rho g(R + h)$; that is, the tension equals the weight of the cable of length $R + h$.

Area and Length

The catenary—i.e., the graph of the hyperbolic cosine—has a remarkable property: the area under the arc above any interval equals its length (see Figure 4). That is,

$$\int_0^x f(t)dt = \int_0^x \sqrt{1 + (f'(t))^2}\,dt \qquad (3)$$

if $f = \cosh$.

One can either check this by substitution or deduce it by differentiating (3) and solving the resulting ordinary differential equation (ODE) for $y = f(x)$:

$$y' = \sqrt{y^2 - 1}, \quad y(0) = 1. \qquad (4)$$

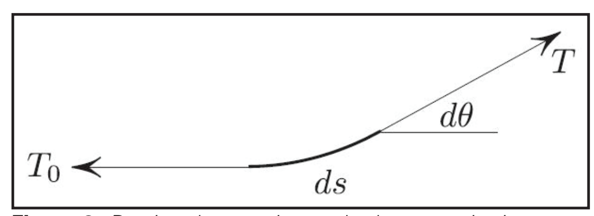

Figure 3. *Proving that tension at the lowest point is proportional to the radius of curvature.*

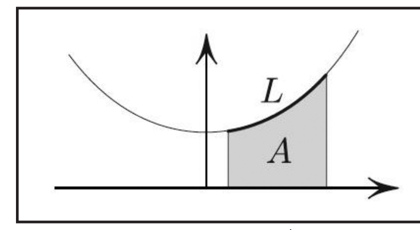

Figure 4. *For the catenary, $A = L$.*

The initial condition $y(0) = 1$ results from substituting $x = 0$ into the derivative of (3). Separation of variables and some manipulation leads to $y = \cosh x$.

Uniqueness

Is cosh the only function with this property? A quick reflection—or a look at (3)—shows that the constant $f(x) \equiv 1$ has the same property. This initially worried me: where did I lose the answer when solving (4)? Having two solutions for the same ODE signals that the uniqueness theorem does not apply; indeed, $\sqrt{y^2 - 1}$ fails assumptions of every uniqueness theorem at $y = 1$. With the two solutions $f(x) = \cosh x$ and $f(x) \equiv 1$, infinitely many others must also exist (according to a theorem of Kneser and Zaremba). However, these solutions are not very interesting; they are simply concatenations that are defined, for an arbitrary c, by $f(x) \equiv 1$ for $x \in [0, c]$ and $f(x) = \cosh(x - c)$ for $x > c$.

The figures in this article were provided by the author.

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