Classroom Notes: Symmetric Matrices and (a Little) Work

MATHEMATICAL CURIOSITIES
By Mark Levi

Here are a few unpretentious observations that occurred to me several years ago after teaching a linear algebra course. I am not making any claim to their originality.

1. The elegant but also a bit antiseptic definition of a symmetric $n \times n$ real matrix A as the one satisfying the identity

$$(A\mathbf{x}, \mathbf{y}) - (\mathbf{x}, A\mathbf{y}) = 0 \quad \text{for all } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n \tag{1}$$

has a physical interpretation: this identity is equivalent to saying that the work done by the linear force field $\mathbf{F}(\mathbf{x}) = A\mathbf{x}$ around the parallelogram generated by $\mathbf{x}$ and $\mathbf{y}$ vanishes (see Figure 1).

Figure 2 illustrates proof of this equivalence. The average forces on each of the sides of the parallelogram are equal to the forces $\mathbf{F}_i$ at the midpoints M_i. The total work W around the parallelogram, grouping parallel sides together, is

$$W = (\mathbf{F}_1 - \mathbf{F}_3, \mathbf{x}) + (\mathbf{F}_2 - \mathbf{F}_4, \mathbf{y});$$

and since $\mathbf{F}_1 - \mathbf{F}_3 = -A\mathbf{y}$ and $\mathbf{F}_2 - \mathbf{F}_4 = A\mathbf{x}$,

this gives $W = (A\mathbf{x}, \mathbf{y}) - (\mathbf{x}, A\mathbf{y})$.

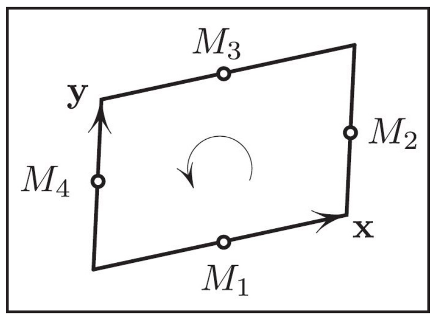

Figure 2. *Physical meaning of (A***x***,* **y***) – (***x***, A***y***).*

In particular, (1) expresses the conservativeness of the vector field $A\mathbf{x}$.

2. Here is a physical reason why eigenvalues of a symmetric matrix are real. Assuming for a moment that they are not, consider the plane spanned by the real and the imaginary parts $\mathbf{u}$, $\mathbf{v}$ of the eigenvector $\mathbf{w} = \mathbf{u} + i\mathbf{v}$. At each point $\mathbf{x}$ in this plane the force $A\mathbf{x}$ lies in the plane (so that we can forget about the rest of $\mathbb{R}^n$). And since the work done by $A\mathbf{x}$ around a circle in this plane - centered around the origin - is zero, the tangential component of $A\mathbf{x}$ changes sign at some point(s) $\mathbf{x}_0$ on the circle – which is to say, $A\mathbf{x}_0$ is normal to the circle at $\mathbf{x}_0$. Thus, $\mathbf{x}_0$ is a (real) eigenvector.

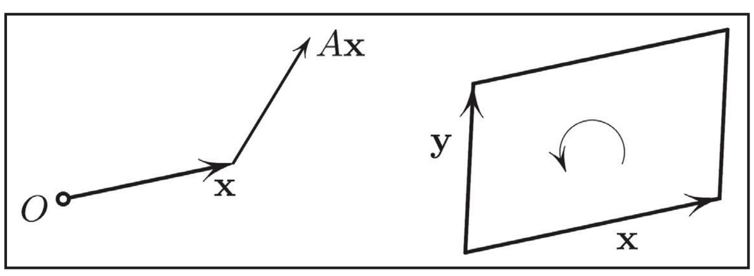

Figure 1. *Vector field A***x** *and the closed parallelogram path.*

3. Orthogonality of the eigenvectors: a physical/geometrical proof. Let $\mathbf{u}$, $\mathbf{v}$ be two distinct eigenvectors of a symmetric $n \times n$ matrix A with the eigenvalues $\lambda \neq \mu = 0$ (the latter assumption involves no loss of generality since we can take $\mu = 0$ by replacing A with $A - \mu I$). Figure 3 shows the force field $A\mathbf{x}$ of such a matrix. Consider the work of $A\mathbf{x}$ around the triangle OQP. The only contribution comes from PO since $A\mathbf{x}$ vanishes along OQ and is normal to QP. And if $A\mathbf{x}$ is conservative, then $W_{PO} = 0$ and hence $P = O$, implying $\mathbf{u} \perp \mathbf{v}$. This completes a "physical" proof of orthogonality of the eigenvectors of symmetric matrices.

4. The entry a_{ij}, $i \neq j$ of a square matrix $A = (a_{ij})$ has a dynamical interpretation: it is the angular velocity, in the (ij)–plane, of $\mathbf{e}_i$ moving with the vector field $A\mathbf{x}$.[1] Indeed, $a_{ij} = (A\mathbf{e}_i, \mathbf{e}_j)$, the projection of the velocity

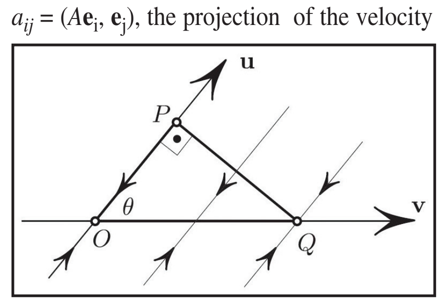

Figure 3. *Geometrical proof of orthogonality.*

$A\mathbf{e}_i$ onto $\mathbf{e}_j$, Figure 4. And thus the symmetry condition $a_{ij} = a_{ji}$, illustrated in Figure 4, also amounts to stating that the 2D curl in every ij–plane vanishes. The symmetry for 3×3 matrices is equivalent to curl $A\mathbf{x} = 0$. In fact, decomposition of a general square matrix into its symmetric and antisymmetric parts amounts to decomposing the

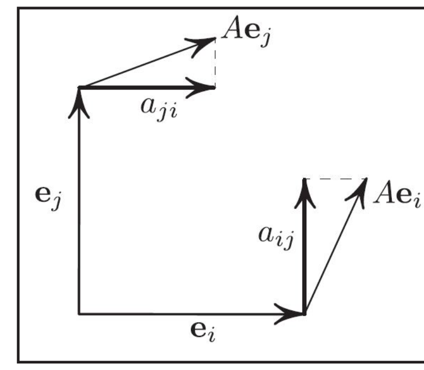

Figure 4. *a_{ij} as the angular velocity.*

vector field $A\mathbf{x}$ into the sum of a curl-free and divergence-free field, a special case of Helmholtz's theorem, itself a special case of the Hodge decomposition theorem.

And the diagonal entries a_{ii} give the rate of elongation of $\mathbf{e}_i$; this explains geometrically why the cube formed at $t = 0$ by $\mathbf{e}_i$ and carried by the velocity field $A\mathbf{x}$ changes its volume at the rate tr A (at $t = 0$). This also offers a geometrical explanation of the matrix identity $\det e^A = e^{\operatorname{tr} A}$.

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1 to be more precise, we should be referring to the moving vector instantaneously aligned with $\mathbf{e}_i$.

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