Perpetual Motion and the Theorem of Cosines

MATHEMATICAL CURIOSITIES

By Mark Levi

What follows is the first installment of a regular column by Mark Levi of the Pennsylvania State University. As proposed a few months ago, the column will consist of "short mathematical/physical morsels which should be of interest to any curious person, including graduate students, and requiring the attention span of a few minutes only . . . and always with pictures." At SIAM News *we liked the idea immediately, based in part on Levi's recent article presenting a newly discovered connection between bicycle tracks and the stationary Schrödinger equation (http://bit.ly/1zk59VF), which was based in turn on his invited talk at SIAM's 2013 conference on dynamical systems (http://bit.ly/12jkd95). We were also favorably impressed to find a category called "Some Nifty Things" on his website.*

The impossibility of creating a perpetual motion machine is a sad fact of life for most people (especially those who are still trying to invent such machines). But this fact has a silver lining: Among other things, it implies some mathematical theorems. As a quick example, here is a derivation of the theorem of cosines.

The "machine" is a rigid triangular container in flatland, free to pivot on a vertex P (Figure 1). As a thought experiment, we fill the triangle with compressed gas. The trapped gas tries to rotate each side of the rigid frame around P. The sum of the three torques is zero (the alternative would be a functioning perpetual motion machine)––and this is the theorem of cosines in disguise. Indeed, deciphering the zero-torque statement we have, according to Figure 1:

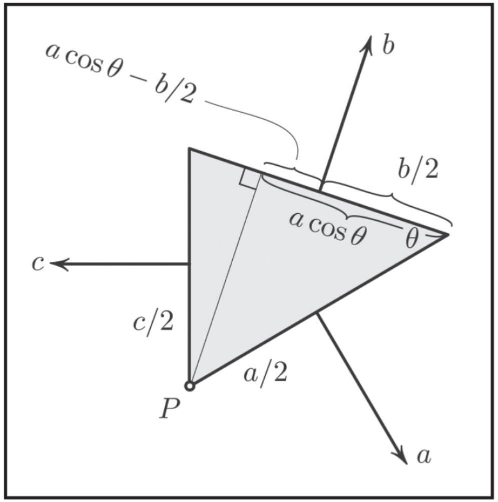


Figure 1. *Proof of the theorem of cosines. Pressure $p = 1$ (in units of force per unit of length), so that the forces are a, b, and c. Equidistributed force on each side was replaced by the force applied at the midpoint.*

$$c\, c/2 + b(a \cos\theta - b/2) = a\, a/2, \quad (1)$$

or, after a quick rearrangement:

$$c^2 = a^2 + b^2 - 2ab \cos\theta.$$

The same idea leads us to discover that for a right triangle we have

$$b^2 - a^2 = 2cd,$$

in the notation of Figure 2.

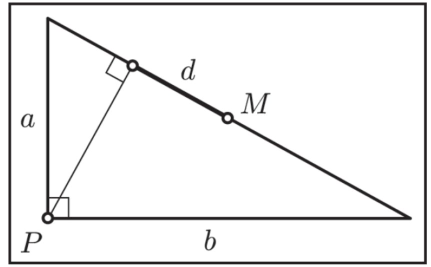
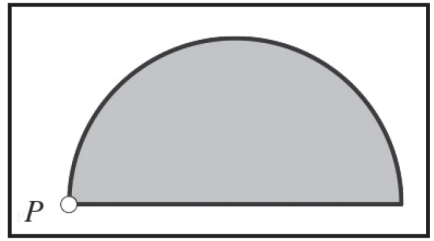


Figure 2. *M is the midpoint of the hypotenuse of length c, and d is the distance from M to the foot of the perpendicular.*

An amusing exercise in the same spirit is to translate the equilibrium statement for the half-disk shown at the right into a formula.



Answer to the exercise:

$$\int_0^{\pi/2} \sin\theta \cos\theta \, d\theta = 1/2$$

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