Minimal Perimeter Triangles

Mark Levi

1. INTRODUCTION. Physics often provides mathematics not only with a problem, but also with the idea of a solution. Physical reasoning has traditionally been a fruitful source of mathematical invention, having played a key role in some of the main discoveries of Archimedes, Bernoulli, and Riemann, among many others. In this article we give a much more modest example of a mathematical result that the author has discovered by means of a physical argument. We will also provide a one-paragraph physical proof of Ceva's theorem used in the discussion (and stated as Theorem 6).

Theorem 1. *Let K be an arbitrary closed convex curve in the plane. Assume, moreover, that the curvature of K is nonzero and finite at every point. From the set of all triangles circumscribed around K consider a triangle $\triangle ABC$ of minimal perimeter.*[1] *Then, referring to Figure 1, the following assertions hold:*

1. *the three perpendiculars to the sides of $\triangle ABC$ at the tangency points are concurrent;*
2. *the three segments connecting a vertex of $\triangle ABC$ with the tangency point on the opposite side of $\triangle ABC$ are concurrent; equivalently, by Ceva's theorem,*

$$a_1 b_1 c_1 = a_2 b_2 c_2, \tag{1}$$

where a_1 and a_2 are the lengths of the segments into which the tangency point divides the side opposite the vertex A, and similarly for b and c.

Remark. After this manuscript was completed, Serge Tabachnikov pointed out to me that L. F. Toth in his book [**4**] has studied polygons of minimal perimeters circumscribed around a closed curve. One of Toth's theorems implies, modulo some additional remarks, statement (2) of Theorem 1. The concurrency of normals in statement (1) turns out to be a much more general fact—it holds for "almost" all functions of

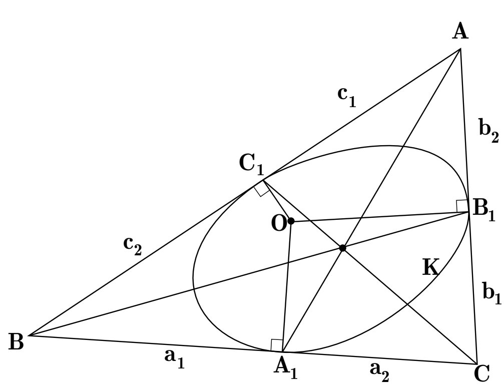

Figure 1. $\triangle ABC$ is a minimal perimeter triangle around K.

[1] "Minimal" can be replaced by "critical."

the triangles, not just the perimeters. For instance, concurrency of normals holds for circumscribed triangles minimizing any of the following: (i) the area; (ii) the sum of the squares $a^2 + b^2 + c^2$ of the side-lengths; (iii) any sum $f(a) + g(b) + h(c)$, where f, g, and h are arbitrary monotone increasing functions. In fact, much more generally, we have:

Theorem 2. *Let $E = E(a, b, c)$ be an aribtrary smooth function of the lengths a, b, and c of the sides of a triangle, and let E satisfy the nondegeneracy assumption $d/d\lambda E(\lambda a, \lambda b, \lambda c)|_{\lambda=1} \equiv \nabla E \cdot \langle a, b, c\rangle \neq 0$.[2] If $\triangle ABC$ minimizes E among all triangles circumscribed around a given closed strictly convex smooth curve K, then the normals to the sides of $\triangle ABC$ at the points of tangency with K are concurrent.*

We mention a few books where physical proofs for other mathematical theorems can be found. A reproduction of Archimedes' discovery of the integral calculus by using mechanics can be found, along with other examples, in chapter 9 of Polya's book [**3**]. Additional proofs based on mechanics can be found in very nice booklets by Uspenskii [**5**] and Kogan [**2**].

2. A PHYSICAL "PROOF." Given a closed convex curve K, we will introduce a mechanical system whose potential energy equals the perimeter of a circumscribed triangle. For such a system

$$\text{minimal energy} \Leftrightarrow \text{minimal perimeter} \Rightarrow \text{equilibrium};$$

the equilibrium condition, properly restated, will give us the conclusion of Theorem 1. The proof of Theorem 2 essentially reduces to the proof of Theorem 1 once the perimeter is replaced with the more general function E in the formulation of Theorem 2.

We describe the mechanical system and then immediately give a proof of the theorem based on the physics of the equilibrium configuration.

Constant tension springs. We will use constant tension springs, i.e., "rubber bands" whose tension is independent of the elongation. Such a spring, depicted in Figure 2, can be realized by creating a vacuum inside a cylinder closed at one end, with a frictionless piston separating the vacuum from the surrounding atmosphere. For convenience we use springs with tension $T = 1$: the potential energy of such a spring equals its length.

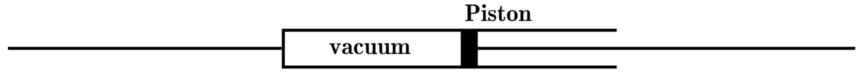

Figure 2. The constant tension spring.

The mechanical system. Consider three (infinite) rods with each pair of rods slipped through a small ring and forming a triangle $\triangle ABC$, as shown in Figure 3. The rods are in frictionless contact with the rings, and thus can form a triangle of any shape, except for the constraint we now impose: $\triangle ABC$ must contain the curve K in its interior, i.e., K is an obstacle impenetrable to the rods. Now let us connect each pair of

[2] This excludes functions E of the angles alone.

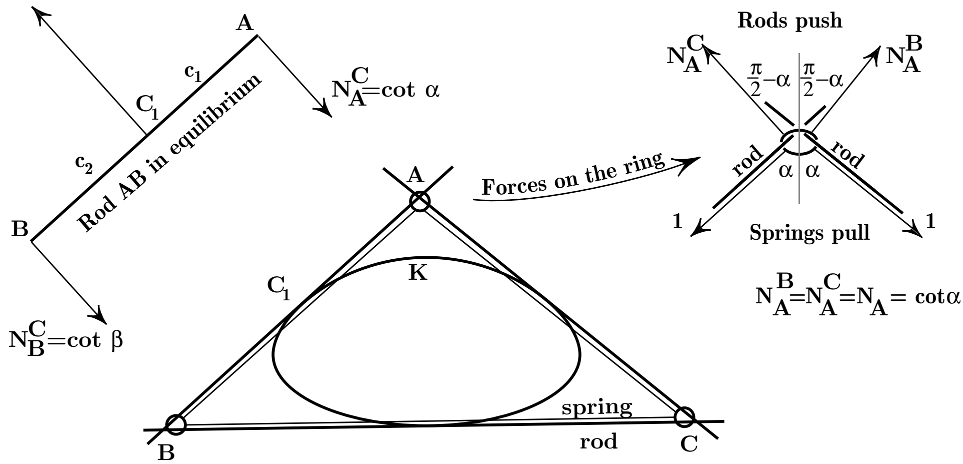

Figure 3. The "springs and rings" system.

rings by a constant tension spring, so that each spring runs along a side of $\triangle ABC$, as shown in Figure 3. We have thus built an "analog computer" that will seek out the minimal perimeter triangle. The springs are trying to collapse $\triangle ABC$, but the obstacle K prevents such collapse. The potential energy of our mechanical system is precisely the perimeter of the triangle formed by the rods. If the perimeter of $\triangle ABC$ is minimal, then so is the energy of our mechanical system, which is thus in equilibrium. Restating the equilibrium condition results in a physical proof of Theorem 1, as follows.

Physical proof of Theorem 1, part (1). Consider the three rods with their rings and springs as one system, subject to three normal reaction forces from K at tangency points. If $\triangle ABC$ has minimal perimeter, then it is in equilibrium, and thus the sum of torques of these forces with respect to any point vanishes. In particular, it vanishes with respect to the point O of intersection of the lines of two of the forces. Thus the line of the third force must also pass through O. ■

Physical proof of Theorem 1, part (2). It suffices to prove that $a_1b_1c_1 = a_2b_2c_2$, as in equation (1). By Ceva's theorem (see [**1**]), the three segments of concern in the theorem (called the *Cevians*) must then be concurrent. We now derive the equilibrium conditions for the rods, which give the desired identity at once.

Consider a minimal perimeter $\triangle ABC$. Each rod is subject to the three normal reaction forces shown in Figure 3 (top left): one force from K and one from each of the two rings that are in contact with the rod. Consider one of the rods, say AB. The normal reaction force from the ring A upon the rod is $\cot\alpha$, where $\alpha = 1/2\angle A$. Indeed, since *all forces acting on the ring A are in balance*, we have $N_A\cos(\pi/2-\alpha) = \cos\alpha$, as in Figure 3 (top right). Here N_A is the magnitude of the force exerted by the rod upon the ring A. Since action equals reaction, N_A is also the force of the ring upon the rod.[3] Similarly, the rings B and C exert forces $\cot\beta$ and $\cot\gamma$, respectively, upon the rods they touch, where $\beta = 1/2\angle B$ and $\gamma = 1/2\angle C$. The zero torque conditions for the three rods, with respect to their points of contact with K, are:

[3] In fact, it is the force upon both rods AB and AC at the points of contact with the ring A.

$$\begin{cases} c_1 \cot\alpha = c_2 \cot\beta, \\ a_1 \cot\beta = a_2 \cot\gamma, \\ b_1 \cot\gamma = b_2 \cot\alpha. \end{cases} \tag{2}$$

Multiplying the equations in (2), we get $a_1 b_1 c_1 = a_2 b_2 c_2$. ■

3. PROOF OF THEOREM 1.

A lemma on variation of the perimeter. We first derive the formula for the derivative of the perimeter of a circumscribed triangle. Let $\triangle ABC$ be an arbitrary (not necessarily minimal) triangle circumscribed around K. We fix two of the straight lines AB and AC and vary the third line BC, parametrizing its point $\mathbf{r} = \mathbf{r}(s)$ of tangency by the counterclockwise arclength s measured along K from a chosen point (Figure 4).

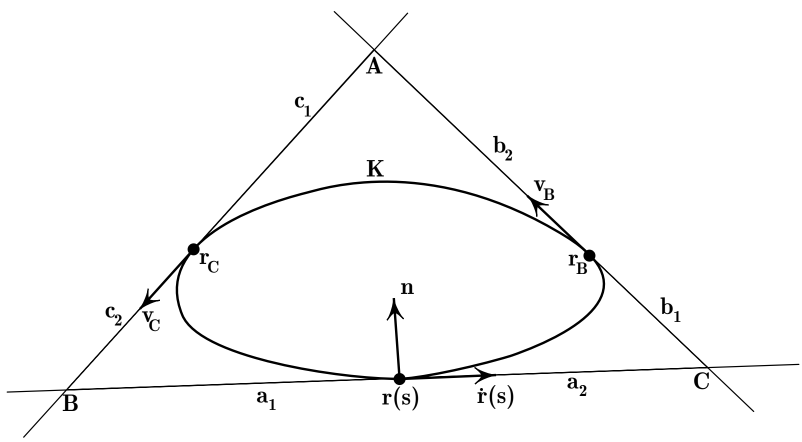

Figure 4. Differentiation of the perimeter.

Lemma 1. *In the notation of Figure 4 and with $|\triangle ABC|$ signifying the perimeter, it is the case that*

$$\frac{d}{ds}(|\triangle ABC|) = (a_1 \cot\beta - a_2 \cot\gamma)k, \tag{3}$$

where $\beta = \beta(s) = 1/2\angle B$, $\gamma = \gamma(s) = 1/2\angle C$, and $k = k(s)$ is the curvature of K at $\mathbf{r}(s)$.

Proof. As s varies, the only relevant segment lengths varying are c_2, a_1, a_2, and b_1, as depicted in Figure 4. The point B lies on two tangent lines to K and hence

$$\mathbf{r}(s) - a_1 \dot{\mathbf{r}}(s) = \mathbf{r}_C + c_2 \mathbf{v}_C,$$

where $\mathbf{r}_C$ is the constant point of tangency of K with the side AB and $\mathbf{v}_C$ is the constant unit tangent vector to K at that point. Differentiation with respect to s gives

$$\dot{\mathbf{r}} - a_1 \ddot{\mathbf{r}} - \dot{a}_1 \dot{\mathbf{r}} = \dot{c}_2 \mathbf{v}_C. \tag{4}$$

We note that $|\dot{\mathbf{r}}| = 1$ and $\ddot{\mathbf{r}} = k\mathbf{n}$, where $\mathbf{n}$ is the unit inward normal vector to K and k is the curvature of K. Projecting each side of (4) first onto $\mathbf{n}$ and then onto $\dot{\mathbf{r}}$ yields

$$-ka_1 = \dot{c}_2 \mathbf{v}_C \cdot \mathbf{n}, \quad 1 - \dot{a}_1 = \dot{c}_2 \mathbf{v}_C \cdot \dot{\mathbf{r}}.$$

Since $\mathbf{v}_C \cdot \mathbf{n} = -\sin\angle B$ and $\mathbf{v}_C \cdot \dot{\mathbf{r}} = -\cos\angle B$, we obtain $\dot{c}_2 = ka_1/\sin\angle B$ and $\dot{a}_1 = 1 + \dot{c}_2 \cos\angle B = 1 + ka_1 \cot\angle B$. Addition gives

$$\frac{d}{ds}(c_2 + a_1) = 1 + ka_1 \left(\frac{1}{\sin\angle B} + \cot\angle B \right) = 1 + ka_1 \cot\beta,$$

with $\beta = 1/2\angle B$. Similarly we obtain (note the minus sign)

$$\frac{d}{ds}(a_2 + b_1) = -(1 + ka_2 \cot\gamma).$$

Adding the last two equations delivers (3):

$$\frac{d}{ds}(|\triangle ABC|) = \frac{d}{ds}(c_2 + a_1 + a_2 + b_1) = (a_1 \cot\beta - a_2 \cot\gamma)k. \qquad \blacksquare$$

Proof of Theorem 1, part (2). Let $\triangle ABC$ be a triangle of minimal perimeter circumscribing K. Allowing the line BC to vary with s as in Figure 4, we have $d/ds(|\triangle ABC|)|_{s=s_0} = 0$, where s_0 is the value of s corresponding to the minimal perimeter. Using (3) we conclude that, in the minimizing configuration,

$$a_1 \cot\beta = a_2 \cot\gamma. \tag{5}$$

Cyclic permutations of this relation give

$$\begin{cases} b_1 \cot\gamma = b_2 \cot\alpha, \\ c_1 \cot\alpha = c_2 \cot\beta. \end{cases} \tag{6}$$

Multiplication of the equations in (5) and (6) then leads to the desired equation (1).

$\blacksquare$

4. PROOF OF THEOREM 2.

The idea of the proof. Let O be the point of intersection of the normals to the sides at the tangency points T_B and T_C (Figure 5). Let us imagine the curve K rotating with unit angular velocity around O, with the triangle undergoing dilation in order to maintain tangency with the curve. We take the moving triangle to coincide with

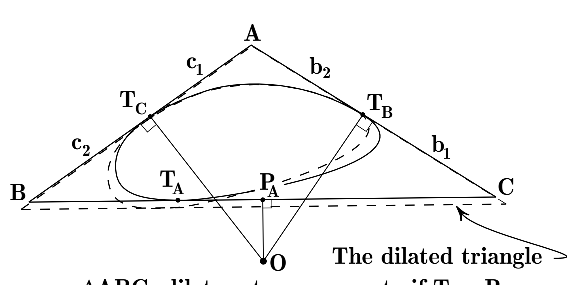

Figure 5. Proof of concurrence of normals.

$\triangle ABC$ at $t = 0$. If $P_A \neq T_A$, where P_A is the foot of the perpendicular from O to BC (as shown in Figure 5), then the triangle has to dilate with nonzero speed to remain tangent to K. This is intuitively obvious but will be proved later. Thus the quantity E changes with nonzero speed, a contradiction. We now make this argument rigorous.

Differentiation with respect to rotations. Instead of rotating the triangle, we will rotate the curve K. Fix an arbitrary point O (later to be chosen as the intersection of two perpendiculars, as described in the previous paragraph). Consider the rotated curve $K_\theta = R_\theta K$, where R_θ is the counterclockwise rotation through the angle θ around O. Fix unit vectors $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$ and consider $\triangle ABC_\theta$ circumscribed around K_θ and having sides parallel to $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$, respectively (Figure 6).[4] The rate at which $\triangle ABC_\theta$ dilates with changing θ is given by the following lemma.

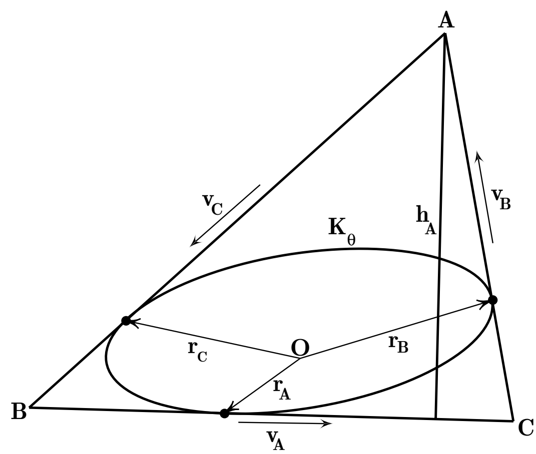

Figure 6. Proof of the lemma on the rate of dilation.

Lemma 2. *The sides a, b, and c of $\triangle ABC_\theta$ depend on θ in such a way that*

$$\frac{d}{d\theta}\langle a, b, c\rangle = \lambda \langle a, b, c\rangle, \tag{7}$$

where, referring to Figure 6,

$$\lambda = -\left(\frac{\mathbf{r}_A \cdot \mathbf{v}_A}{h_A} + \frac{\mathbf{r}_B \cdot \mathbf{v}_B}{h_B} + \frac{\mathbf{r}_C \cdot \mathbf{v}_C}{h_C}\right). \tag{8}$$

Here $\mathbf{r}_A$ is the position vector (relative to O) of the tangency point of the side BC, and the vectors $\mathbf{r}_B$ and $\mathbf{r}_C$ are defined similarly. In other words, as the curve rotates, the triangle undergoes the instantaneous dilation at the exponential rate λ.

Remark. The expression λ is independent of the choice of O. Indeed, letting $\mathcal{A} = \text{area}(\triangle ABC)$, we substitute $h_A = 2\mathcal{A}/a$, $h_B = 2\mathcal{A}/b$, and $h_C = 2\mathcal{A}/c$ into (8) obtaining $\lambda = -(\mathbf{r}_A \cdot \mathbf{a} + \mathbf{r}_B \cdot \mathbf{b} + \mathbf{r}_C \cdot \mathbf{c})/2\mathcal{A}$, where $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors that represent the sides of $\triangle ABC$ in the orientation suggested by Figure 6. Moving O to a new center $O' = O + \mathbf{k}$ results in the new $\lambda' = \lambda + \frac{1}{2\mathcal{A}}\mathbf{k} \cdot (\mathbf{a} + \mathbf{b} + \mathbf{c}) = \lambda$ (because

[4] There are actually two such triangles for each θ; we choose the one whose vertices A, B, and C occur counterclockwise.

$\mathbf{a} + \mathbf{b} + \mathbf{c} = \mathbf{0}$), as claimed. We postpone the proof of Lemma 2, first applying it to complete the proof of Theorem 2.

Proof of Theorem 2. Let $\triangle ABC$ minimize $E(a, b, c)$ over all triangles that circumscribe K. Let the center of rotation O be the intersection of normals at tangency points T_B and T_C (see Figure 5). With O thus specified, consider $\triangle ABC_\theta$ as defined in the foregoing discussion. Using equation (7), we obtain from the minimality of E for the given triangle:

$$0 = \frac{d}{d\theta} E(a, b, c)|_{\theta=0} = \lambda \nabla E \cdot \langle a, b, c \rangle. \tag{9}$$

We conclude that $\lambda = 0$, for the dot product does not vanish because of the nondegeneracy assumption on E. Now λ is given by (8), where the last two terms vanish by our choice of O; hence, $(\mathbf{r}_A \cdot \mathbf{v_A})/h_A = 0$. ■

Proof of Lemma 2. First let us consider the family of all triangles (unrelated to K) with sides parallel to $\mathbf{v}_A$, $\mathbf{v}_B$, and $\mathbf{v}_C$, and treat the distances x, y, and z from O to the sides of $\triangle ABC$ as independent variables, so that each of the side-lengths a, b, and c is a function of x, y and z (Figure 7).[5] Since *all the triangles in this family are similar*, regardless of the choice of x, y, and z, we note that the ratio $a(x, y, z)/h_A(x, y, z)$ is constant. Logarithmic differentiation with respect to x gives

$$\frac{1}{a}\frac{\partial a}{\partial x} = \frac{1}{h_A}\frac{\partial h_A}{\partial x}.$$

Since $\partial h_A/\partial x = 1$, we get

$$\frac{1}{a}\frac{\partial a}{\partial x} = \frac{1}{h_A}.$$

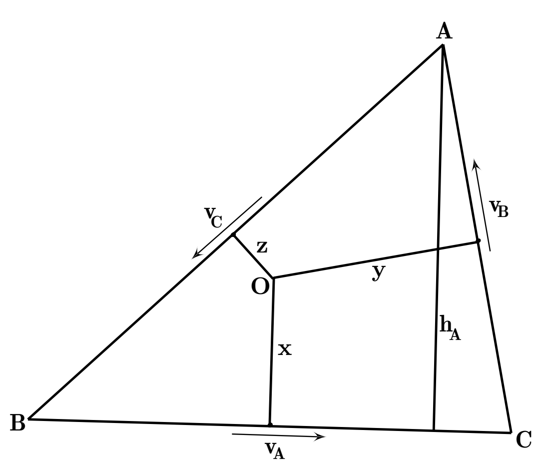

Figure 7. Triangles parametrized by x, y, and z.

[5]To be more precise, each of the quantities x, y, and z is the *directed* distance from O to the corresponding side of $\triangle ABC$ *in the direction of the outward normal*. If O happens to lie outside $\triangle ABC$, then one or two of the distances x, y, and z will be negative.

Similarly, logarithmic differentiation with respect to x of the constants b/h_A and c/h_A gives

$$\frac{1}{b}\frac{\partial b}{\partial x} = \frac{1}{h_A} \quad \text{and} \quad \frac{1}{c}\frac{\partial c}{\partial x} = \frac{1}{h_A},$$

whence

$$\frac{\partial}{\partial x}\langle a, b, c\rangle = \frac{1}{h_A}\langle a, b, c\rangle. \tag{10}$$

Similarly

$$\frac{\partial}{\partial y}\langle a, b, c\rangle = \frac{1}{h_B}\langle a, b, c\rangle, \quad \frac{\partial}{\partial z}\langle a, b, c\rangle = \frac{1}{h_C}\langle a, b, c\rangle. \tag{11}$$

Now restrict attention to the triangles $\triangle ABC_\theta$ circumscribed around K_θ. Then x, y, and z all depend on a single variable θ, and from Lemma 3 (to follow) we obtain

$$\frac{dx}{d\theta} = -\mathbf{r}_A \cdot \mathbf{v}_A, \quad \frac{dy}{d\theta} = -\mathbf{r}_B \cdot \mathbf{v}_B, \quad \frac{dz}{d\theta} = -\mathbf{r}_C \cdot \mathbf{v}_C. \tag{12}$$

Using the chain rule, we infer from (10), (11), and (12) that

$$\frac{d}{d\theta}\langle a, b, c\rangle = \frac{\partial}{\partial x}\langle a, b, c\rangle\frac{dx}{d\theta} + \frac{\partial}{\partial y}\langle a, b, c\rangle\frac{dy}{d\theta} + \frac{\partial}{\partial z}\langle a, b, c\rangle\frac{dz}{d\theta} = \lambda\langle a, b, c\rangle,$$

with $\lambda = -(\mathbf{r}_A \cdot \mathbf{v_A}/h_A + \mathbf{r}_B \cdot \mathbf{v_B}/h_B + \mathbf{r}_C \cdot \mathbf{v_C}/h_C)$. ∎

In the course of the preceding proof we made appeal to Lemma 3. Here is the result in question.

Lemma 3. *Consider the curve K_θ obtained by rotating K through the angle θ around a given point O (Figure 8). Let* $\mathbf{v}$ *be a fixed unit vector in the plane, and consider the distance $x(\theta)$ from O to a tangent line to K_θ in the direction of* $\mathbf{v}$*. Then*

$$x'(0) = -\mathbf{r}(s_0) \cdot \mathbf{v}, \tag{13}$$

where s_0 is such that $\mathbf{r}'(s_0) = \mathbf{v}$.

Proof. We have

$$x(\theta) = (R_\theta \mathbf{r}(s_\theta)) \cdot \mathbf{n}, \tag{14}$$

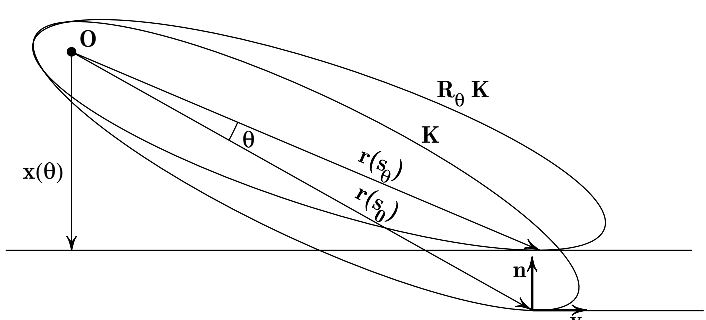

Figure 8. The distance to a rotating curve.

where s_θ is defined by the parallel tangent condition $R_\theta \mathbf{r}'(s_\theta) = \mathbf{v}$ and $\mathbf{n} = R_{\pi/2}\mathbf{v}$ is the unit vector normal to $\mathbf{v}$. The parallel tangent condition, rewritten as

$$R_\theta \mathbf{r}'(s_\theta) \cdot \mathbf{n} = 0,$$

defines s_θ as a smooth function of θ for θ near 0 (This is actually true for all θ, but we do not need that information). This is geometrically obvious; more formally, the equation $f(s,\theta) \equiv R_\theta \mathbf{r}'(s) \cdot \mathbf{n} = 0$ defines s as a function of θ since the conditions of the implicit function theorem hold: $f(s_0, 0) = 0$ by the definition of s_0 and $\partial/\partial s f(s_0, 0) = \mathbf{r}''(s_0) \cdot \mathbf{n} = k(s_0)\mathbf{n} \cdot \mathbf{n} = k(s_0) \neq 0$. In particular, s_θ is a smooth function of θ.

Differentiating (14) at $\theta = 0$, we obtain

$$x'(0) = \frac{d}{d\theta}(R_\theta \mathbf{r}(s_0))|_{\theta=0} \cdot \mathbf{n} + \frac{ds_\theta}{d\theta}|_{\theta=0}\mathbf{r}'(s_0) \cdot \mathbf{n} = R_{\frac{\pi}{2}}\mathbf{r} \cdot \mathbf{n} = -\mathbf{r} \cdot \mathbf{v},$$

where we have used $d/d\theta R_\theta|_{\theta=0} = R_{\pi/2}$, $\mathbf{r}'(s_0) \cdot \mathbf{n} \equiv \mathbf{v} \cdot \mathbf{n} = 0$, and $R^{-1}_{\pi/2}\mathbf{n} = -\mathbf{v}$. This establishes (13). ■

5. A PHYSICAL PROOF OF CEVA'S THEOREM We refer to [**1**] for a nice treatment of Ceva's theorem and for further references. Here our goal is to give a *physical* "proof."

Theorem 3 (Ceva's theorem and its converse). *Given a triangle $\triangle ABC$, consider three segments AA_1, BB_1, and CC_1, where A_1, B_1, and C_1 divide the sides opposite the vertices A, B, and C into lengths denoted by a_1 and a_2, b_1 and b_2, and c_1 and c_2, respectively (labeled as in Figure 9). The three segments are concurrent if and only if $a_1b_1c_1 = a_2b_2c_2$.*

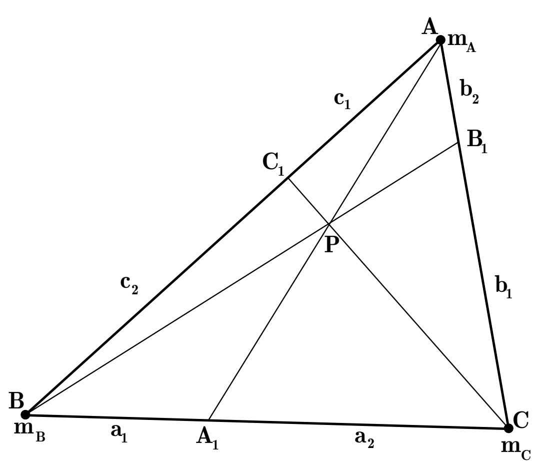

Figure 9. Ceva's theorem by mechanics.

Proof. Assume that the three segments AA_1, BB_1, and CC_1 shown in Figure 9 share a common point P. Let us endow the vertices A, B, and C with masses m_A, m_B, and m_C, chosen so that their center of mass (c.m.) lies at P. (To that end, first choose $m_B = a_2$ and $m_C = a_1$, thereby placing c.m.(B, C) at A_1. Then choose m_A to ensure that c.m.$(A, A_1) = P$ by setting $m_A = t(m_B + m_C)$, where t is the ratio of the length of PA_1 to the length of AP.)

To complete the proof of the necessity part of the theorem, it remains to observe that A_1, B_1, and C_1 are the centers of mass of their respective sides. Indeed, *the center of mass of three point masses lies on the line through one mass and the center of mass of the other two.* In particular, the line through A and c.m.(B, C) contains P. But line AA_1 contains P as well; hence $A_1 =$ c.m.(B, C). The same argument holds for the other two sides.

Since $A_1 =$ c.m.(B, C), $B_1 =$ c.m.(C, A), and $C_1 =$ c.m.(A, B), we obtain

$$m_B a_1 = m_C a_2, \quad m_C b_1 = m_A b_2, \quad m_A c_1 = m_B c_2.$$

Multiplying these three equations we obtain the desired identity (1).

The converse (that relation (1) implies concurrency) is easily proved by contradiction, as in [**1**]. Given that equation (1) holds, we assume that one of the pertinent segments, say CC_1, does not pass through the intersection of the other two. A *different* segment $C\tilde{C}_1$ with $\tilde{C}$ on AB but $\tilde{C} \neq C$ does pass through the intersection, and the last identity applies: $a_1 b_1 \tilde{c}_1 = a_2 b_2 \tilde{c}_2$. But this is in contradiction with (1), since $\tilde{c}_2/\tilde{c}_1 \neq c_2/c_1$. The proof is complete. ∎

ACKNOWLEDGMENTS. Partial support by the NSF grant DMS-9704554 is gratefully acknowledged. It is a pleasure to thank Dima Burago and Pavel Bleher for helpful suggestions for simplifying proofs, and Serge Tabachnikov for bringing the result of Toth to my attention. I am also grateful to the referee for constructive criticisms that helped to improve the exposition.

REFERENCES

1. H. S. M. Coxeter and S. L. Greitzer, *Geometry Revisited*, Mathematical Association of America, Washington, D.C., 1967.
2. B. Iu. Kogan, *The Applications of Mechanics to Geometry*, University of Chicago Press, Chicago, 1974.
3. G. Polya, *Mathematics and Plausible Reasoning*, vol. 1, Princeton University Press, Princeton, 1990.
4. L. F. Toth, *Lagerungen in der Ebene auf der Kugel und im Raum*, Springer-Verlag, Berlin, 1953.
5. V. A. Uspenskii, *Some Applications of Mechanics to Mathematics*, Pergamon Press, New York, 1961.

MARK LEVI is a professor in the Department of Mathematics at the Pennsylvania State University. He received his Ph.D. from the Courant Institute under the direction of Jürgen Moser. His research interests include differential equations and dynamical systems with applications to physical problems. One of his hobbies is creating and collecting physical devices that he uses to illustrate mathematical phenomena. This collection includes a 7-foot cycloidal trough used to demonstrate three remarkable properties of the cycloid discovered by Huygens over three hundred years ago. His other hobby is inventing and collecting physical solutions to mathematical problems.
Department of Mathematics, The Pennsylvania State University, University Park, PA 16802
levi@math.psu.edu