Some Shallow Observations

MATHEMATICAL CURIOSITIES

By Mark Levi

Water in a clear pool is deeper than it looks. What seems like five feet when looking straight down is actually seven feet — an unpleasant surprise for a non-swimmer. And this disparity between the true and apparent depths is quite striking for objects that are viewed at an angle. For example, when I direct my gaze 30 degrees below the horizontal, the true depth is more than five times greater than what I see!

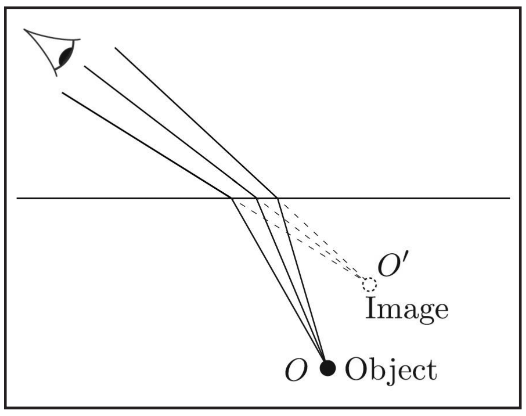

Figure 1. *The image of the object is the point from which the rays that reach the eye seem to emanate, at least to the leading order of accuracy. In reality, each dotted line is tangent to a caustic (see Figure 4) and it only looks like they come out of one point.*

Location of the Image

The image O' of the object O in Figure 1 is the apparent source of rays, i.e., the point from which the rays that reach the eye *seem* to emanate. To find the location of O' given O, let us advance the crossing point C in Figure 2 by a small distance dx, causing small changes in α and β. To the leading order of approximation, the length of the thick segment in Figure 2b can be expressed in two ways:

$$L\,d\beta = dx\cos\beta.$$

In a similar way, $L'd\alpha = dx\cos\alpha$ and thus

$$\frac{L'}{L} = \frac{\cos\alpha}{\cos\beta}\cdot\frac{d\beta}{d\alpha}. \qquad (1)$$

To find $d\beta/d\alpha$, we differentiate Snell's law $\sin\alpha/c_{\text{air}} = \sin\beta/c_{\text{water}}$ to obtain

$$\frac{d\beta}{d\alpha} = c\frac{\cos\alpha}{\cos\beta},$$

where

$$c = c_{\text{water}}/c_{\text{air}} \approx 0.7,$$

the reciprocal of the refraction index. Substituting this into (1) results in the nice expression

$$\frac{L'}{L} = c\frac{\cos^2\alpha}{\cos^2\beta} \qquad (2)$$

for the ratio of the apparent length to the true length. *This is the factor by which the underwater part of a tilted stick seems to shorten.*[1] Note that this shortening becomes drastic at oblique angles; for $\alpha \uparrow \pi/2$, we have $L'/L \to 0$.

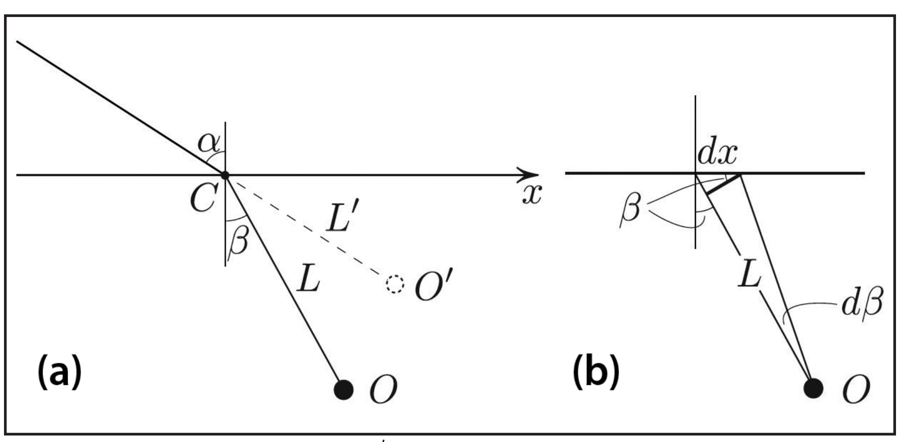

Figure 2. *Computing the location of O'—an infinitesimal move of C by dx—results in infinitesimal changes to α and β.*

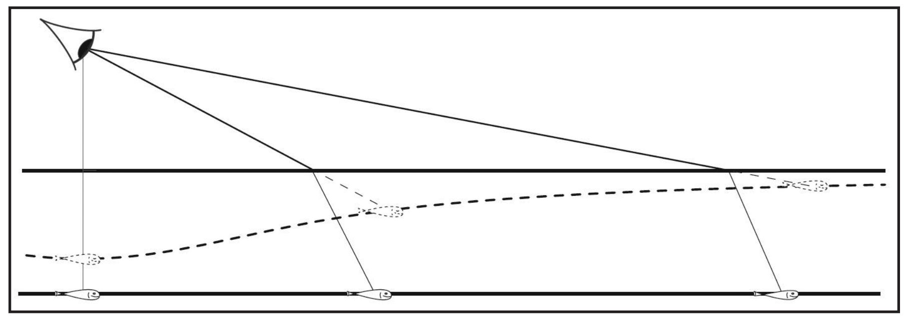

Figure 3. *The image (dotted) of a straight log on the bottom of the lake is an inverted bell-shaped curve whose asymptote is the surface of the water.*

Apparent Depth Versus True Depth

This ratio is given by an elegant expression that we obtain by dividing the apparent depth $L'\cos\alpha$ by the true depth $L\cos\beta$ and using (2):

$$\frac{\text{apparent depth}}{\text{true depth}} = c\frac{\cos^3\alpha}{\cos^3\beta},$$

where c is as before. For example, the above ratio for $\alpha = \pi/3$ is $< 1/5$, a surprisingly small number; the water is more than five times deeper than it seems at that angle.

Distorted Images

Straight lines underwater do not look straight to the above observer (see Figure 3). A "dual" problem that may interest some students is to find the shape of the bottom whose image is straight (for the position of the eye fixed).

The Path of the Image

When the object O is fixed, all of its possible images for all viewers form a cusp (see Figure 4). As an observer swings around—as shown in the figure—the image travels along the caustic. The cusp's shape varies homothetically with the depth of O.

More Questions

With the eye's position fixed, the map $\varphi := O \mapsto O'$ of the lower half plane onto itself is well defined. It would be interesting to describe precisely how this map distorts small objects. That is, can we say something nice about the Jacobian of φ? What is its polar decomposition, for example? For my purposes here, however, this is going too deep.

The figures in this article were provided by the author.

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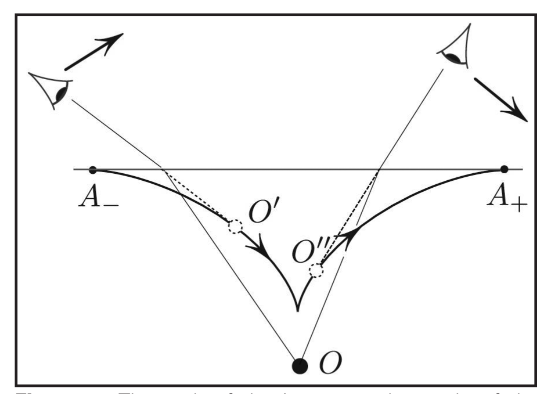

Figure 4. *The path of the image as the angle of the incoming ray into the viewer's eye changes. This cusped path is the envelope of the family of the above-water rays that originate at O. As the eye swings from left to right, the image travels from A_- to A_+.*

[1] Although it should be noted that the image of a straight stick is not quite straight, albeit not noticeably so.