Geometric Sum, Geometrically

MATHEMATICAL CURIOSITIES
By Mark Levi

It is customary to prove the geometric sum formula

$$1+\lambda+\lambda^2+\ldots+\lambda^{n-1}=\frac{1-\lambda^n}{1-\lambda} \quad (1)$$

by algebra. But a geometric sum deserves a geometric treatment, so here are some geometrical proofs.

Proof by Dilation

Let us subject the segment $[0,1]$ to a linear dilation $x \mapsto \lambda x$ that is repeated n times—as shown in Figure 1—for positive $\lambda<1$. The iterates $\lambda,\ldots,\lambda^n$ break $[0,1]$ into $n+1$ subintervals. And the iterates of the rightmost interval $[\lambda,1]$ have geometrically decreasing lengths $\lambda^i(1-\lambda)$ with $i=0,\ldots,n-1$, as Figure 1 shows. The combined length of all the intervals, recording from right to left in the figure, is 1:

$$(1-\lambda)+\lambda(1-\lambda)+\ldots+\lambda^{n-1}(1-\lambda)+\lambda^n=1,$$

thus resulting in (1).

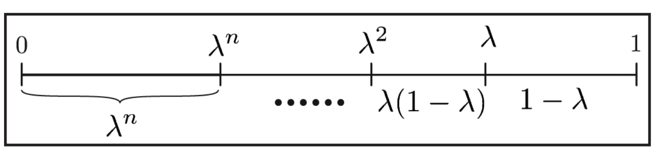

Figure 1. *Proof of (1) by dilation.*

The remaining two proofs are for the infinite sum

$$1+\lambda+\lambda^2+\ldots=\frac{1}{1-\lambda} \quad (2)$$

and positive $\lambda<1$.

A Pythagorean Proof

The construction of Figure 2 yields a partition of the hypotenuse into an infinite union of segments of lengths 1, $\sin^2\theta, \sin^4\theta,\ldots$ On the other hand, the hypotenuse has length $(1/\cos\theta)/\cos\theta=\cos^{-2}\theta$, so that

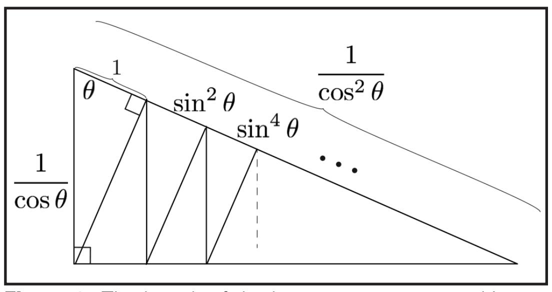

Figure 2. *The length of the hypotenuse computed in two different ways yields (3).*

$$1+\sin^2\theta+\sin^4\theta+\ldots=\frac{1}{\cos^2\theta}. \quad (3)$$

This implies (2) by choosing θ so that $\lambda=\sin^2\theta$, and by application of the Pythagorean theorem. As a curiosity, reversing the argument—i.e., taking (2) for granted—gives an admittedly strange proof of the Pythagorean theorem.

A Staircase Proof

The two lines $y=1+\lambda x$ and $y=x$, which appear in Figure 1, intersect at height $y=\frac{1}{1-\lambda}$. But this height is also the sum of rises that form a geometric sequence 1, λ, $\lambda^2,\ldots,$ again yielding (2).

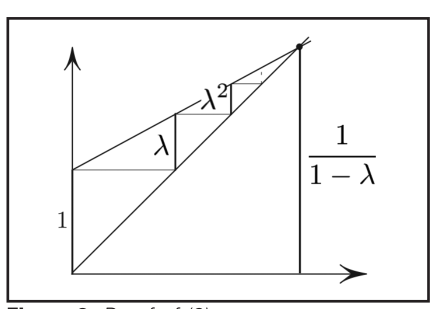

Figure 3. *Proof of (2).*

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