SIAM REVIEW
Vol. 30, No. 4, December 1988

009

STABILITY OF THE INVERTED PENDULUM—A TOPOLOGICAL EXPLANATION*

MARK LEVI†

Abstract. An explanation and a proof of stability of the inverted pendulum whose suspension point undergoes vertical periodic oscillations is given. The main idea of the argument is topological; as it turns out, existence of stable regimes can be proven with little effort using only very crude qualitative information about the system. More precisely, let n be the number of times the pendulum becomes vertical during one forcing period. If n changes by more than 4 with the change of a parameter μ, then for an open interval of intermediate values of μ the pendulum will be stable.

Key words. stability, symplectic group

AMS(MOS) subject classifications. 34F, 58F

Introduction. A well-known surprising result in mechanics states that an inverted pendulum with vertically oscillating suspension point is stable for certain parameter values[1]. There are numerous papers and textbooks on this classical problem [2], [5]–[12], [16] with all of the analyses based on calculations. In this note we give a simple topological proof and explanation of this stability phenomenon. Our explanation is based on the observation that the set of stable 2×2 matrices "obstructs" the solid torus formed by all 2×2 matrices of determinant 1. In physical terms, this will be shown to imply that if the number of oscillations of the pendulum during one period of forcing changes by at least 1 with the change of a parameter, then for some intermediate interval of parameter values the pendulum will actually be stable.

We will omit the detailed description of standard analytic facts about stability of linear Hamiltonian systems with one degree of freedom, such as Mathieu equation; these facts can be found in many textbooks [2], [3], [14], although we do give the background necessary to make the exposition self-contained.

1. Equations of motion. The small angle motions near the top are governed by

$$l\ddot{\phi}-(g+a(t,\mu))\phi=0, \tag{1}$$

when $g>0$ is the acceleration due to gravity and $a(t, \mu)$ is the acceleration of the suspension point of the pendulum given by $a(t, \mu) = \ddot{h}(t, \mu)$, where $h(t, \mu)$ is a time-periodic height function $h(t+1, \mu) = h(l, \mu)$ and μ is a parameter. Equation (1) can be written as a system:

$$\frac{d}{dt}\begin{pmatrix}\phi\\ \psi\end{pmatrix}=\begin{pmatrix}0 & 1/l\\ -g-a & 0\end{pmatrix}\begin{pmatrix}\phi\\ \psi\end{pmatrix}, \tag{2}$$

or in the matrix form:

$$\dot{x}=A(t)x, \tag{2'}$$

where $A(t)$ is a time-periodic matrix, and $x = \binom{\phi}{\psi}$.

* Received by the editors May 10, 1987; accepted for publication September 16, 1987.

† Mathematics Department, Boston University, Boston, Massachusetts 02215. This research was partly supported by the Air Force Office of Scientific Research.

[1] In fact, the classical theory of the Mathieu equation $\ddot{x} + (a + b \sin t)x = 0$ can be interpreted in this light: parts of stability regions lie in the left half-plane $a < 0$ [16] (cf. (1)).

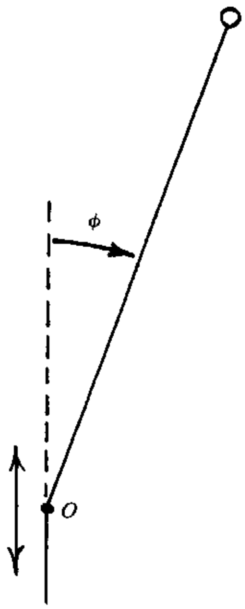

FIG. 1. *A frictionless hinge attaches the rod of the pendulum to the vertically oscillating suspension point O.*

2. Background: Floquet theory. In this section we recall that *stability of* (2) *is equivalent to the stability of the associated Floquet matrix*; below we define all the concepts mentioned in this sentence.

We shall say that the pendulum is (linearly) *stable* if and only if every solution of (1), or equivalently (2), stays bounded for all times $-\infty < t < \infty$; that is, if for any solution $x(t) = \binom{\phi}{\dot\phi}$ of (2) there exists a constant C such that $|x(t)| < C$ for all t with $|x|$ denoting (say) the Euclidean norm.

We can formulate the stability of (2) in terms of its *Poincaré map*, i.e., the map which assigns to each vector $x_0 \in \mathbf{R}^2$ the vector x_1 obtained by following the solution of (2), over one period, starting at x_0. Linearity of the system implies linearity of its Poincaré map, which can therefore be represented by a matrix M, called the *Floquet matrix*.

The Floquet matrix can be constructed out of solution vectors of (2) as follows. Fix two independent (i.e., noncollinear) solutions $x_1(t)$ and $x_2(t)$ of (2). Any other solution $x(t)$ is a linear combination $x(t) = c_1x_1(t) + c_2x_2(t)$, or in matrix form: $x(t) = X(t)c$, where X is the matrix whose columns are the vectors $x_1(t)$ and $x_2(t)$, and $c = \text{col}\,(c_1, c_2)$. Choosing the solutions $x_1(t), x_2(t)$ to start at the unit coordinate vectors e_1, e_2 at $t = 0$, we obtain $X(0) = I$, the identity matrix, and thus

$$x(t) = X(t)x(0).$$

In particular, $x(1) = X(1)x(0)$, i.e., $X(1) \equiv M$ is the Floquet matrix.

A matrix M is called *stable* if all of its integer powers are bounded: there exists $c > 0: |M^n| < c$ for any integer n. Here $|\ |$ denotes any matrix norm, say the maximum norm.

The pendulum equation (2) *is stable if and only if its Floquet matrix M is stable.*

Indeed, if (2) is stable, then the matrix $|X(t)|$ is bounded for all t, and in particular, the set of $X(n) = M^n$ is bounded for any integer n, thus proving the stability of M. Conversely, if M is stable, i.e., $|X(n)| = |M^n| < c$, then for any $0 \leqq \tau < 1$

we have $|X(t)| = |X(n+\tau)| = |X(\tau)X(n)| = |X(\tau)M^n| < c \max_{0<\tau<1} |X(\tau)| < \infty$. Consequently, any solution $x(t) = X(t)x(0)$ is bounded, i.e., (2) is stable.

$X(t)$ is called the *fundamental matrix*, or the *fundamental solution*. The Hamiltonian character of (2) manifests itself in a special property: $\det X = 1$—this is a consequence of the relationship $d/dt \det X = \operatorname{tr} A \det X = 0$. The last relation says that the rate of change of the unit area in the phase plane of (2) is given by the divergence of the vectorfield.

The set of all matrices of determinant 1 is denoted Sp (1) (in general Sp (n) denotes the set of all symplectic $n \times n$ matrices). We shall think of $X(t)$, $0 \leqq t \leqq 1$ as a curve in Sp (1) (Fig. 2); stability of (2) depends on whether the endpoint $M = X(1)$ of the curve $X(t)$ lies in the subset $S \subset$ Sp (1) consisting of stable matrices. Understanding the topology of these two sets, Sp (1) and S, is thus necessary to understanding the problem at hand.

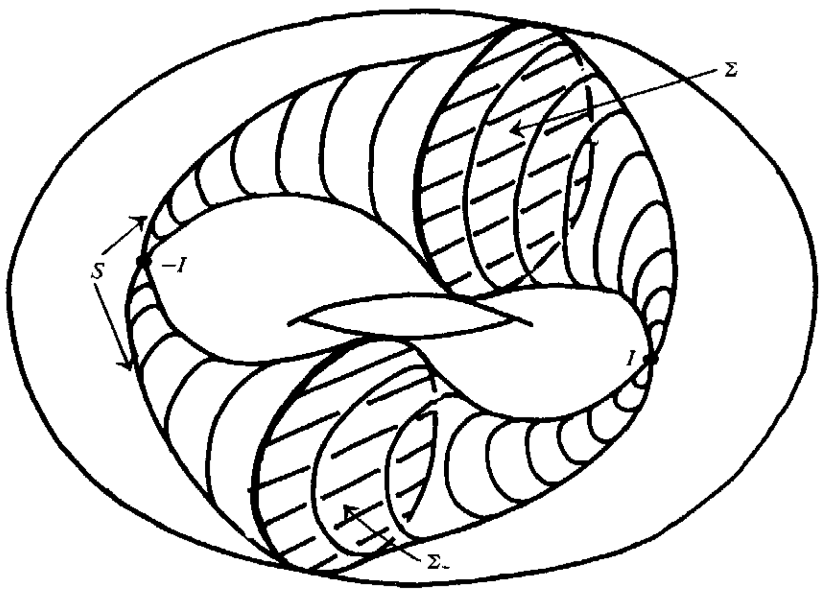

FIG. 2. Sp (2) *is an open solid torus. Both components of the set S consisting of stable matrices obstruct* Sp (1), *since both surfaces* Σ *and* Σ_- *do.*

3. Topology of Sp (1).

LEMMA. *The set* Sp (1) *is topologically an open solid three-dimensional torus. The subset S obstructs the torus; more precisely, any noncontractible loop in* Sp (1) *must intersect S (Fig. 2).*

The first statement of the lemma is due to Gelfand and Lidskii [1]. To prove the lemma, we note that any matrix M with $\det M = 1$ admits a unique polar factorization $M = PQ$, where $P^T = P$ is positive definite with $\det P = 1$, and $Q^T = Q^{-1}$ is a rotation matrix. (To prove this, we define $P = (MM^T)^{1/2}$, where the positive definite value of the square root matrix is chosen, and sets $Q = P^{-1}M$.) It is clear that P and Q depend on M continuously. The set of all positive definite $P \in$ Sp (1) forms an *open* disk, while the set of all rotation matrices

$$Q = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

is a circle. Consequently, Sp (1) is an open solid torus.

To show that the stable set S obstructs this torus, it suffices to find a two-dimensional surface Σ in S consisting of stable matrices, which obstructs Sp (1) (Fig. 2). Letting $R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ be the $\pi/2$-rotation, we set

$$\Sigma = \{PR, P^T = P > 0, \det P = 1\}.$$

Clearly, the "meridianal plane" Σ obstructs the torus Sp (1), since P above ranges over the entire open disc. Furthermore, any matrix PR from Σ is stable. Indeed, let $v_1, v_2, \lambda_1, \lambda_2$ be the normalized eigenvectors and the eigenvalues of P. Since $v_1 \perp v_2$, we have $Rv_1 = v_2$, $Rv_2 = -v_1$, and hence, using $\lambda_1\lambda_2 = 1$ we obtain

$$(PR)^2 v_1 = PRPv_2 = \lambda_2 PRv_2 = -\lambda_2 Pv_1 = \lambda_1\lambda_2 v_1 = -v_1,$$

and similarly, $(PR)^2 v_2 = -v_2$. We conclude that $(PR)^2 = -I$ and thus all powers of PR are bounded. The argument above shows also that any matrix of the form PR is similar to a $\pi/2$-rotation.

We observe that another obstructing surface Σ_- is obtained by replacing R by $-R$, the rotation by $-\pi/2$ (see Fig. 2). As a side remark, we point out that any matrix of the form PR is *strongly stable*, i.e., lies in the interior of the set S.

4. Explanation of stability of the inverted pendulum via the torus obstruction. Here is the gist of our argument. Assume that (2) depends on a parameter μ in such a way that as μ changes from μ_1 to μ_2, the Floquet matrix $M = M(\mu)$ travels once around the torus Sp (1); but then M will necessarily have to dip into (both components of) the stable set S for some intermediate values of μ; for these values the pendulum will be stable. This explains the phenomenon in principle, but the question remains: how does the revolution of M around the hole in the torus manifest itself in the change in physical motion of the pendulum? The answer is, *roughly*, this: one revolution of M corresponds to the change by one of the number of oscillations around the top equilibrium that the pendulum makes during one forcing period. We make this more precise by assigning the argument ("angle") to the Floquet matrix (cf. [1]). We define the argument function $\alpha(t)$ along the curve $X(t)$ of matrices as the angle α of the rotational part $Q(t)$ of $X(t)$ in its polar factorization. We choose $\alpha(0) = 0$, corresponding to $X(0) = I$, and furthermore, by insisting on the continuity of $\alpha(t)$ we remove the ambiguity in the choice of the angle. The argument of $M = M(\mu)$ is now defined as a single-valued function $\alpha = \alpha(t, \mu)_{t=1}$. Physically, α corresponds to the number of oscillations of the pendulum during one period of forcing. More precisely, we have the following theorem.

THEOREM. *The number of times the pendulum becomes vertical during one period is between $n - 1$ and $n + 3$, where $n = [|\alpha|/\pi]$ is the net integer number of half-turns the matrix $X(t)$ makes around the hole in* Sp (1). *In other words, the number of zeros of any nontrivial solution $\phi(t)$ of* (1) *in the interval $0 \leqq t < 1$ is between $n - 1$ and $n + 3$.*

Proof. Let $x = \binom{\phi}{\psi}$ be a solution of (2). We have

$$x(t) = X(t)x(0) = P(t)Q(t)x(0);$$

by definition of n the matrix $X(t)$ makes n full half-turns around the torus, and hence the vector $Q(t)x(0)$ makes n full half-turns around the origin in the plane. Now, the angle between the vectors PQc and Qc never exceeds $\pi/2$ due to the positive definite symmetric character of P, and thus the vector $x(t) = PQx(0)$ turns by at least $\pi n - \pi/2$ and at most by $(n + 1)\pi + \pi/2$. Thus $\phi(t)$ has between $n - 1$ and $n + 3$ roots.

COROLLARY. *If for two distinct values of the parameter μ the numbers of times the pendulum passes the top equilibrium during one period differ at least by* 4, *then for some intermediate μ the pendulum is stable.*

Example. Consider the pendulum equation (1) with the acceleration of the suspension point alternately $+2g$ and $-2g$ up and down, i.e., $a(t)=2g \operatorname{sgn} \sin 2\pi t$. With $1/l \equiv \mu$ chosen as the parameter, (1) takes the form

$$\ddot{\phi}-\mu(g+a(t))\phi=0. \tag{3}$$

As the length l of the pendulum tends to zero, the number of roots of any solution ϕ during $0<t<1$ tends to ∞. Indeed, for the first half period the flow of (3) is a hyperbolic rotation, and for the second half it is an elliptic rotation through the angle $\sim\sqrt{\mu}$. By the corollary above there exist infinitely many intervals of μ for which the pendulum is stable.

Remark. So far we have not addressed the question of stability of the more precise nonlinear model of the pendulum:

$$l\ddot{\alpha}-(g+a(t,\mu))\sin\alpha=0. \tag{4}$$

Equation (1) is the linearization of (4) around the top equilibrium solution $\alpha=0$. Such an approximation is extremely accurate for small α; however, it is possible that even a small error could build up over a long time thus destroying stability. It is a remarkable consequence of the KAM theory that under some generic nonlinearity conditions the errors do not build up over the *infinite* time interval. Geometrically, let $\Phi:(\alpha,\dot{\alpha})_{t=0}\rightarrow(\alpha,\dot{\alpha})_{t=1}$ be the Poincaré map of (4); the origin $\alpha=\dot{\alpha}=0$ is the fixed point. The question is whether this fixed point is stable—more precisely, whether for any $\varepsilon>0$ there exists $\delta>0$ such that all iterates of any point from the δ-neighborhood of the origin remain in the ε-neighborhood. Stability of (1) is equivalent to stability of the derivative $M=d\Phi(0,0)$. Now, Kolmogorov–Arnaud–Moser (KAM) theory [17] gives the answer to the above nonlinear stability question. More precisely, if an eigenvalue λ of $d\Phi$ satisfies the nonresonance condition $\lambda^3\neq 1$, $\lambda^4=1$ and if the first Birkhoff invariant does not vanish (i.e., the twist is present in the cubic truncation of the normal form), then the origin is surrounded by (a Cantor set worth of) invariant curves and thus is stable (see [17]). Generically, these conditions are violated for at most a discrete set of μ. Any solution of (4) starting on such an invariant curve is quasiperiodic. Since the relative measure of invariant curves tends to one near the origin, the small amplitude motions will be quasiperiodic with probability close to 1.

It is interesting to observe that the above nonresonance conditions on λ have a geometrical illustration in terms of the solid torus Sp (1) in Fig. 2. The matrices with $\lambda=i$, for instance, form exactly the union of "obstructing membranes" $\Sigma\cup\Sigma_-$. The nonresonance conditions require that the Floquet matrix $M=d\Phi$ of the linearized equation (1) not land on any of the six obstructing disks corresponding to $\lambda=\pm1$, $\pm i$, $-\frac{1}{2}\pm\sqrt{3}/2$.

REFERENCES

[1] I. M. GELFAND AND L. D. LIDSKII, *On the structure of stability of linear Hamiltonian systems of differential equations with periodic coefficients*, Uspekhi Mat. Nauk, 10 (1955), pp. 3–40. (In Russian.) AMS Transl. (2) 8 (1958), pp. 143–181. (In English.).

[2] V. I. ARNOLD, *Geometric Theory of the Ordinary Differential Equations*, Springer-Verlag, Berlin, New York, 1984.

[3] CODDINGTON AND LEVINSON, *Theory of Ordinary Differential Equations*, McGraw-Hill, New York, 1955.

[4] J. B. KELLER AND M. I. WEINSTEIN, SIAM J. Appl. Math., 45 (1985), pp. 200–214.
[5] H. C. CORBEN AND P. STEHLE, *Classical Mechanics*, second edition, John Wiley, New York, 1960, pp. 67–69.
[6] D. J. NESS, *Small oscillations of a stable inverted pendulum*, Amer. J. Phys., 35 (1967), p. 964.
[7] L. BLITZER, *Inverted pendulum*, Amer. J. Phys., 33 (1965), p. 1076.
[8] F. M. PHELPS III AND J. H. HUNTER, JR., *An analytic solution of the inverted pendulum*, Amer. J. Phys., 33 (1965), pp. 285–295.
[9] H. P. KALMUS, *The inverted pendulum*, Amer. J. Phys., 38 (1970), p. 874.
[10] R. E. KING, *The inverted pendulum*, Amer. J. Phys., 33 (1965), p. 855.
[11] A. GRAY AND J. GRAY, *A Treatise on Dynamics*, second edition, Macmillan, New York, 1920, pp. 414, 425.
[12] N. MINORSKY, *Mechanical Vibrations*, second edition, McGraw–Hill, New York, 1940.
[13] B. VAN DER POL AND STRUTT, *On the stability of Mathieu equation*, Philos. Math., 5 (1928), p. 18.
[14] N. W. MCLACHLEN, *Theory and Applications of Mathieu Functions*, Oxford University Press, London, 1947; Dover, New York, 1964, 2.10.
[15] J. WALKER, *The Flying Circus of Physics With Answers*, John Wiley, New York, 1977.
[16] J. J. STOKER, *Theory of Vibrations*, Interscience, New York, London.
[17] C. L. SIEGEL AND J. K. MOSER, *Lectures on Celestial Mechanics*, Springer-Verlag, New York, 1971.