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Geometry and physics of averaging with applications

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Abstract

The main point of this paper is an observation that behind the standard averaging procedure there lies some simple previously unobserved geometry. In particular, the averaged forces in a rapidly forced system are, as it turns out, the constraint forces of an associated auxiliary non-holonomic system. The curvature of these constraints enters the expression for the averaged system. For example, the curvature of the pursuit curve enters the averaged equation of the pendulum with vibrating suspension point. This observation gives a new physical and geometrical insight into the mechanics of the Paul traps and the stability of forced inverted pendula. ©1999 Elsevier Science B.V. All rights reserved.

Keywords: Paul trap; Averaging; Normal form; Vibration; Constraints

1. Introduction

This note contains a new observation on the well-studied problem of Hamiltonian systems with rapid time-dependence.

This observation gives a new insight into the methods of confinement of charged particles such as the Paul trap [1,2], Fig. 1. As a side remark, a very simple physical explanation of stability of the inverted pendulum with vertically vibrating suspension is given.

The recent discovery of stable π-kinks in rapidly forced sine-Gordon equations [3] is based on the same idea.

1.1. The paper's outline and related applications

Plan of the paper. In Section 2, we state the main theorem and its corollaries for two particular cases: the unit force field, where the averaging theory gives a purely geometrical answer, and for potential fields which arise in most applications.

In Section 3, we discuss applications (A) to the Paul trap; (B) to the forced multiple pendulum (with a brief derivation of the averaged Lagrangian formulation, slightly more general than Theorem 2); (C) to the single pendulum, (D) to Hill's equation and finally (E) to composition of non-commuting symplectic matrices.

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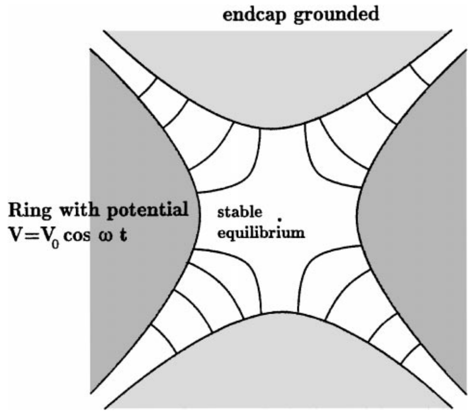

Fig. 1. Schematics of the Paul trap.

Section 4 contains an informal ‘proof’ of Theorem 1; a proof is given in Section 5.

We conclude this introduction with the mention of physical manifestations of the phenomenon analyzed in this note.

Early history. The stabilizing effect of vibrations has been known as early as 1908 through Stephenson's [4] experimental demonstration of stability of an inverted pendulum with the vertically oscillating suspension. Somewhat later, in 1928, van der Pol and Strutt [5] have drawn the stability diagram for the Mathieu's equation. This diagram implies, among other facts, Stephenson's result. In 1951, Kapitsa [6,7] studied stability of the inverted pendulum through effective potentials [8] and had also suggested to apply vibrational stabilization to mechanical objects other than pendula, such as large molecules.

The Paul trap. The related idea of levitating charged particles via an oscillating electric field (the ‘Paul trap’) goes back to 1958 [1,9]; for this work Paul was awarded the Nobel Prize in 1989. See also [2,10,11]. The Paul trap consists of three electrodes: a ring and two endcaps, Fig. 1. The endcaps are grounded, while a time–periodic potential is applied to the ring. Since the electrostatic potential is a harmonic function, it has no minima, and thus no stable equilibria are possible in an electrostatic field. The remarkable fact is that the net effect of the high frequency vibrations is to create a minimum for the *effective* potential [8]. This paper gives a new explanation of the stabilization mechanism in the Paul trap. The standard explanation (or rather calculation) of stability [9] is based on linearizing near the equilibrium and on verifying that the resulting Mathieu-type linear system is stable.

Strong focusing in synchrotrons. The discovery of the Paul trap was preceded by the idea of strong focusing in synchrotrons. This idea, patented by Christofilos [12], was rediscovered by Courant, Livingston, Snyder and Blewett [13] (who suggested the term ‘strong focusing’) in 1952, see [14]. Strong focusing of particles in synchrotrons is accomplished by placing a series of closely spaced magnetic lenses in the path of the beam of particles. The lenses are alternately focusing and defocusing in one of the transversal directions; the average effect of this alternation is the focusing in both transversal directions. Here the space variation plays the role of rapid time-dependence in the Paul trap.

Dipoles with spins. We mention in this connection the beautiful theory of the motion of a magnetic dipole possessing the mechanical angular momentum in a magnetic field, see [15–20] and references therein. Averaging over rapid rotations of such a dipole produces an additional Lorentz-type force as well as an effective electrostatic force, see [15]. The latter force can be interpreted in terms of curvature; this will be shown in a future publication. The Lorentz force mentioned above is partly responsible for stability of the ‘Levitron™’ – a magnetic top which hovers for several minutes, when spun properly, over another constant magnet. For stability analysis of the ‘Levitron™’, we refer to recent papers [19–21].

2. A theorem on averaging and its corollaries

In this paper, we consider the motion of a particle $\boldsymbol{x} \in \boldsymbol{R}^n$ subject to a time-dependent force field of special form:

$$\ddot{\boldsymbol{x}} = a(t, \varepsilon)\boldsymbol{f}(\boldsymbol{x}); \tag{1}$$

we assume the scalar 'acceleration' a to be rapidly oscillating:

$$a(t, \varepsilon) = \varepsilon^{\alpha} A\left(\frac{t}{\varepsilon}\right), \quad \text{with} \quad \alpha > -2 \tag{2}$$

where $A(\tau + 1) = A(\tau) = \mathrm{O}(1)$ and $\varepsilon \ll 1$. We note that

- the acceleration is allowed to be very large, but
- the 'displacement' $s(t, \epsilon)$ must be small:

$$s(t, \varepsilon) = \int_{t_0}^{t}\int_{\tau_0}^{\tau} (a(s) - \langle a\rangle)\, \mathrm{d}s\, \mathrm{d}\tau \sim \varepsilon^{2+\alpha} \ll 1; \tag{3}$$

here $t_0,\ \tau_0$ are chosen so that the integral is a periodic function with zero average.

We are thus dealing with high-frequency violent vibrations. Throughout the paper, we assume that $\boldsymbol{f}$ is real analytic ($\boldsymbol{C}^5$ actually suffices).

Theorem 1. *Under the scaling assumption* (2), *the system* (1) *reduces to the averaged equation*

$$\ddot{\boldsymbol{X}} = \langle a\rangle \boldsymbol{f}(\boldsymbol{X}) - \langle v^2\rangle \boldsymbol{f}'(\boldsymbol{X})\boldsymbol{f}(\boldsymbol{X}) + \boldsymbol{E}, \tag{4}$$

where [1]

$$v = \int_{t_0}^{t} (a - \langle a\rangle)\, \mathrm{d}t, \quad \langle v\rangle = 0 \tag{5}$$

and where $|\boldsymbol{E}| \le M\varepsilon^{3+5\alpha/2}$, *via the transformation* $x = X + s(t, \varepsilon) f(X) + \cdots$, *where* s *was defined in Eq.* (3)*; the remaining terms of the transformation are given in the proof of the theorem* (*Section* 5.3).

We note that $\boldsymbol{f}'\boldsymbol{f}$ has the interpretation of the acceleration of the particle moving in the velocity field $\boldsymbol{f}$. Eq. (4) has several interesting physical implications which are explored below.

Proof of the theorem is given in Section 5; alternatively, one can obtain the proof from a similar result for the first order systems, cf. the book by Bogoliubov and Mitropolski [22]. We have made no attempt to generalize this theorem; on the contrary, the simplest nontrivial case (1) was chosen to demonstrate the phenomenon in the clearest possible way.

2.1. An application to unit vectorfields

In this case, the geometry becomes particularly transparent:

Corollary 1 (The unit vectorfield). *If* $|\boldsymbol{f}(\boldsymbol{x})| \equiv 1$, *then Eq.* (4) *becomes*

$$\ddot{\boldsymbol{X}} = \langle a\rangle \boldsymbol{f}(\boldsymbol{X}) - k\langle v^2\rangle \boldsymbol{n} + \boldsymbol{E}, \tag{6}$$

where $\boldsymbol{n}(\boldsymbol{x})$ *is the principal unit normal vector to the integral curve of* $\boldsymbol{f}$ *at* $\boldsymbol{x}$ *and where* $k(\boldsymbol{x})$ *is the curvature of this curve*. $\boldsymbol{E}$ *satisfies the same estimate as above*.

[1] $\boldsymbol{f}'(\boldsymbol{X})$ denotes the Jacobian $n \times n$ matrix of partial derivatives. In the solid mechanics literature, the expression $\boldsymbol{f}'\boldsymbol{f}$, called the convective derivative, is written as $(\boldsymbol{f} \cdot \nabla)\boldsymbol{f}$.

Observe that $kv^2\boldsymbol{n}$ is precisely the average centrifugal force acting on a bead of unit mass sliding along a curve of curvature k with speed v.

Proof of the corollary follows at once from Eq. (4) by application of

Lemma 1. *In a vector field of unit vectors: $|\boldsymbol{f}(\boldsymbol{x})| = 1, \forall \boldsymbol{x}$, the curvature $k = k(\boldsymbol{x})$ and the principal unit normal $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{x})$ to the integral line through $\boldsymbol{x}$ are given by*

$$k\boldsymbol{n} = \boldsymbol{f}'\boldsymbol{f} \equiv (\boldsymbol{f} \cdot \nabla)\boldsymbol{f}. \tag{7}$$

Proof of Lemma 1. Let $\boldsymbol{x} = \boldsymbol{x}(s)$ be an integral curve: $\mathrm{d}\boldsymbol{x}(s)/\mathrm{d}s = \boldsymbol{f}(\boldsymbol{x}(s))$; since $|\boldsymbol{f}| \equiv 1$, s is the arc length parameter. Thus

$$k\boldsymbol{n} = \frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}s^2} = \frac{\mathrm{d}}{\mathrm{d}s}\boldsymbol{f}(\boldsymbol{x}(s)) = \boldsymbol{f}'(\boldsymbol{x})\boldsymbol{f}(\boldsymbol{x}).$$

This completes the proof of the lemma and of Corollary 1. □

In the special case $\boldsymbol{x} \in \boldsymbol{R}^2$, the last corollary can be restated as follows:

Corollary 2. *For the unit vectorfield in $\boldsymbol{R}^2$, the averaged truncated equations take the form*

$$\ddot{\boldsymbol{X}} = \langle a \rangle \boldsymbol{f}(\boldsymbol{X}) - \langle v^2 \rangle \operatorname{curl} \boldsymbol{f} \cdot \boldsymbol{n}, \tag{8}$$

where curl denotes the 2*-dimensional (scalar) curl*.

This follows from

Lemma 2. *The curvature $k(\boldsymbol{x})$ of trajectories of a unit vectorfield $\boldsymbol{f}(\boldsymbol{x}) \equiv 1$ in $\boldsymbol{R}^2$ coincides with the (scalar) curl*:

$$\operatorname{curl} \boldsymbol{f}(\boldsymbol{x}) = k(\boldsymbol{x}), \tag{9}$$

where the sign of k is decided by the definition of the curvature: $k = \mathrm{d}\theta/\mathrm{d}s$, with the direction of increasing s given by $\boldsymbol{f}$.

Proof of Lemma 2. By the definition, the curl in 2D is the sum of the angular velocities of two infinitesimal segments A and B orthogonal to each other at $\boldsymbol{x}$ as these segments are carried with the flow: $\operatorname{curl} \boldsymbol{f} = \omega(A) + \omega(B)$. Take one of these segments A to be tangent to the trajectory at $\boldsymbol{x}$, so that $B \perp A$ at $\boldsymbol{x}$ at the initial instant, and let both A and B be carried along with the flow. Note that $\omega(A) = k$ by the definition of the curvature: indeed, A remains tangent to the orbit for all t as it is carried with the flow with the speed $|\boldsymbol{f}| = 1$. On the other hand, we have $\omega(B) = 0$ at $t = 0$ since $\boldsymbol{f} \perp B$ and since $|\boldsymbol{f}| = \text{const}$. We conclude that $\omega(A) + \omega(B) = k + 0 = k$, Q.E.D. □

2.2. An application to potential fields

If the force field is potential, such as in the Paul trap, then the effective force is also potential:

Theorem 2 (Potential force). *If $\boldsymbol{f} = -\nabla V(\boldsymbol{x})$ in Eq.* (1)[2] *so that the latter becomes*

$$\ddot{\boldsymbol{x}} = -a\nabla V, \tag{10}$$

[2] We use the notations $V'(\boldsymbol{x}) \equiv \nabla(\boldsymbol{x})$ interchangeably.

then the averaged Eq. (4) *becomes, assuming for simplicity* $\langle a\rangle = 0$:

$$\ddot{\boldsymbol{X}} = -\langle v^2\rangle\nabla W + \boldsymbol{E}, \quad \textit{where} \quad W = \tfrac{1}{2}\boldsymbol{f}^2 = \tfrac{1}{2}(\nabla V)^2. \tag{11}$$

Proof of Theorem 2. $\boldsymbol{f}'\boldsymbol{f} = \frac{1}{2}(\boldsymbol{f}^2)' \equiv \nabla W$, Q.E.D. □

Remark. *According to this theorem, every isolated equilibrium in* V *is a minimum of* W. *Every isolated equilibrium of the original potential system is thus stabilized in the sense that every such equilibrium point lies in a well of a time-independent potential* W, *and in addition is a subject to a rapidly oscillating perturbation force* $\boldsymbol{E}$. *Note that this force is indeed small compared to the leading term in Eq.* (11): $|\boldsymbol{E}| \approx \varepsilon^{1+(\alpha/2)}\langle v^2\rangle \ll \langle v^2\rangle$. *The net effect is the attraction towards the equilibrium.*

According to Earnshaw's theorem, stable equilibrium of a rigid object made of charges and masses cannot be achieved in a static field with a harmonic potential, such as any combination of electrostatic, magnetic and gravitational fields. One sees thus that such stabilization becomes possible, as in the Paul trap, if the field oscillates.

Error bounds and stability. The above discussion leaves open the question of stability of Eq. (11) for all time. In fact, such stability is virtually certain to be violated due to Arnold diffusion. For the case of $n = 1$, the proof of stability for all time depends on the verification of the assumptions of the KAM theory. By now there is a considerable literature dealing with such questions, see [23] and references therein. Using the methods developed there, one can prove stability under some mild additional assumptions on V.

We now return to the general case $n \geq 1$. Let $\boldsymbol{X}$ and $\bar{\boldsymbol{X}}$ be the solutions of Eq. (11) and of the truncated equation respectively, sharing the same initial condition. The error $\boldsymbol{X} - \bar{\boldsymbol{X}}$ remains small for time $\mathrm{O}(\varepsilon^{-\alpha/2})$, as follows from results in [24] applied to the system in Section 5. For $\alpha < 0$ ('violent' vibration) this time is short and such estimates are of little use. Nevertheless, one can still obtain physically useful estimates on the energy for longer time intervals. Take, to be specific, the case of $\alpha = -1$ so that

$$\boldsymbol{E} = \mathrm{O}(\sqrt{\varepsilon}) \equiv \sqrt{\varepsilon}\boldsymbol{G}\left(\boldsymbol{x}, \dot{\boldsymbol{x}}, \frac{t}{\varepsilon}, \varepsilon\right), \tag{12}$$

where $\boldsymbol{G}$ is bounded when its arguments are.

Theorem 3. *Fix initial conditions* $\boldsymbol{x}_0$, $\dot{\boldsymbol{x}}_0$, *and let* $\boldsymbol{x}$ *be the solution of Eq.* (11). *Consider the case*[3] *of* $\alpha = -1$. *There exist positive constants* ε_0 *and* K *independent of* ε *such that the 'energy'*

$$\mathcal{H} = \tfrac{1}{2}(\dot{\boldsymbol{x}} - v(t)\boldsymbol{f}(\boldsymbol{x}))^2 + W(\boldsymbol{x})$$

satisfies

$$|\mathcal{H}(t) - \mathcal{H}(0)| \leq \sqrt{\varepsilon}Kt + K\varepsilon, \quad \textit{for} \quad |t| < \frac{1}{K\sqrt{\varepsilon}}. \tag{13}$$

This estimate shows that the particle spends a long time in the well of the effective potential W.

Proof of the Theorem 3. Let $\boldsymbol{X}$ be the solution of the equivalent system (11) which corresponds to $\boldsymbol{x}$ via transformations of Section 5. As long as $(\boldsymbol{X}, \dot{\boldsymbol{X}})$ stays in a compact set of $\boldsymbol{R}^{2n}$, we have $\boldsymbol{x} = \boldsymbol{X} + \mathrm{O}(\varepsilon)$ and $\dot{\boldsymbol{x}} = \dot{\boldsymbol{X}} + v(t)\boldsymbol{f}(\boldsymbol{X}) + \mathrm{O}(\varepsilon)$; here $|\mathrm{O}(\varepsilon)| < c\varepsilon$, with c depending on the compact set mentioned above and independent of ε.

[3] for simplicity; the general case goes verbatim.

To the solution $\boldsymbol{X}$, we associate the energy $H(t) \equiv H(\boldsymbol{X}, \dot{\boldsymbol{X}}) = (1/2)\dot{\boldsymbol{X}}^2 + \langle v^2 \rangle W(\boldsymbol{X})$; according to the above, we have

$$\mathcal{H}(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = H(\boldsymbol{X}, \dot{\boldsymbol{X}}) + \mathrm{O}(\varepsilon),$$

with $|\mathrm{O}(\varepsilon)| < c\varepsilon$, and the theorem follows from the estimate (14) below. This completes the proof of the theorem modulo the following lemma. □

Lemma 3. *Fix the initial condition* $(\boldsymbol{X}_0, \dot{\boldsymbol{X}}_0)$ *and consider the corresponding solution of Eq.* (11), *where we take* $\alpha = -1$. *There exist positive constants* ε_0 *and* K *independent of* ε *such that* $H(t) \equiv H(\boldsymbol{X}, \dot{\boldsymbol{X}}) = (\dot{\boldsymbol{X}}/2) + \langle v^2 \rangle W(\boldsymbol{X})$ *satisfies*

$$|H(t) - H(0)| \le \sqrt{\varepsilon} K t, \quad for \quad |t| < \frac{1}{K\sqrt{\varepsilon}} \tag{14}$$

for any $0 < \varepsilon < \varepsilon_0$.

Proof of Lemma 3. Let $\boldsymbol{X}$ be the solution of Eq. (11) with the initial data $(\boldsymbol{X}_0, \dot{\boldsymbol{X}}_0)$. Let $t^* > 0$ be the first moment when $|H(t^*) - H(0)| = 1$. For all t in the interval $0 \le t \le t^*$, we have $|H(t)| \le |H(0)| + 1 \equiv H_1$, implying $|\dot{\boldsymbol{X}}| < \sqrt{2H_1}$. Differentiating H and using first Eq. (11), then Eq. (12) and finally the last bound on $|\dot{\boldsymbol{X}}|$ yields

$$|\dot{H}| = |\dot{\boldsymbol{X}}(\ddot{\boldsymbol{X}} + \langle v^2 \rangle \nabla W)| = |\sqrt{\varepsilon}\dot{\boldsymbol{X}}\boldsymbol{G}| < \sqrt{\varepsilon}\sqrt{2}H_1 G_M \equiv \sqrt{\varepsilon}K,$$

where G_M is the bound on $\boldsymbol{G}$ over the set $\{(\boldsymbol{X}, \dot{\boldsymbol{X}}) \in \boldsymbol{R}^{2n} : H \le H_1\}$. We conclude that $|H(t) - H(0)| \le K\sqrt{\varepsilon}t$ for all $0 \le t \le t^*$; it remains to show that $t^* = \mathrm{O}(1/\sqrt{\varepsilon})$. The definition of t^* implies $1 = |H(t^*) - H(0)| \le \sqrt{\varepsilon}Kt^*$ (the last inequality was just proven), giving $t^* \ge 1/K\sqrt{\varepsilon}$. This completes the proof of the lemma and thus of Theorem 3. □

It should be mentioned that the main theorem and the above estimates can be strengthened by averaging the higher order terms; we do not do this here. Finally, we mention that to prove stability of the *linearization* of system (11), one must verify the parametric nonresonance conditions in the limit of small ε. This question is not addressed here.

3. Applications

3.1. The Paul trap

The particle in the Paul trap is governed by Eq. (10), where a is proportional to qV; here q is the charge and $V(x)$ is the electrostatic potential in the cavity created by unit voltage difference between the electrodes, Fig. 1.

We now give a heuristic explanation of stabilization. Consider a charged particle in the field of the Paul trap, as in Fig. 1. A more detailed view shown in Fig. 2 suggests that the particle tends to drift, due to its inertia, in the direction of the convexity of the curve. Since the convexity is towards the saddle point, so is the direction of the drift (Fig. 1). This intuitive stabilization argument is not complete: it does not explain why the saddle attracts from *every* direction. A precise geometrical explanation of this effect is given in [25].

3.2. Double pendulum

The additional equilibria. Consider the double pendulum whose suspension point undergoes periodic vertical oscillations. The equations of motion are in the form slightly more general than Eq. (1). They come from

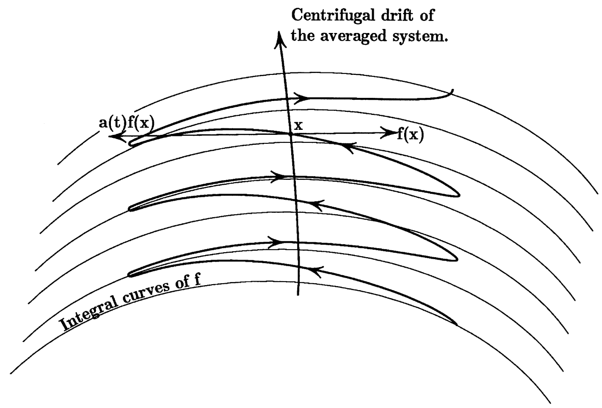

Fig. 2. The particle drift is governed by curvature.

the Lagrangian

$$L(\boldsymbol{x}, \dot{\boldsymbol{x}}, t) = \tfrac{1}{2}\langle \boldsymbol{A}(\boldsymbol{x})\dot{\boldsymbol{x}}, \dot{\boldsymbol{x}}\rangle - a\left(\frac{t}{\varepsilon}\right) V(\boldsymbol{x}), \tag{15}$$

where $\boldsymbol{A}$ is a positive definite symmetric matrix, with $\boldsymbol{x} = (\theta_1, \theta_2)$, here θ_1 and θ_2 are the deflection angles from the vertical, and where $\langle\cdot,\cdot\rangle$ denotes the dot product. Here $a = -g+$acceleration of the suspension point; assuming the latter to be large, we treat g as a small perturbation, omitting it altogether. This omission has no effect on the qualitative picture. We thus consider Eq. (15) with $\langle a\rangle = 0$.

The averaged equations have the Lagrangian (modulo higher order terms)

$$\mathcal{L}(\boldsymbol{X}, \dot{\boldsymbol{X}}, t) = \tfrac{1}{2}\langle \boldsymbol{A}(\boldsymbol{X})\dot{\boldsymbol{X}}, \dot{\boldsymbol{X}}\rangle - \langle v^2\rangle W(\boldsymbol{X}), \tag{16}$$

where $W = (1/2)\langle(\boldsymbol{A}^{-1}V', V'\rangle$. The potential W is a function on a two-dimensional torus, and we wish to count critical points of W which correspond to the equilibria of the truncated averaged system. Since V has four critical points $\boldsymbol{x} = (\theta_1, \theta_2) = (\pi k_1, \pi k_2)$, where $k_i \in \{0, 1\}$. W has four minima. A function on the torus with four minima must have other critical points, so that we know without any computation that there must be additional equilibrium positions for the double pendulum. We now determine the least number of these equilibria by a topological argument.

Let m_0, m_1 and m_2 be the respective numbers of minima, saddles and maxima of W. The Euler characteristic of the torus is $0 = m_0 - m_1 + m_2$. Since $m_0 =$ the number of critical points of V, we have $m_0 \geq 4$ (we assume the generic case of nondegenerate critical points). Since W must have at least one maximum, we have $m_2 \geq 1$. From this, we conclude that $m_1 = m_0 + m_2 \geq 4+1 = 5$, and thus the number of equilibria of W is at least 10. For the case of a double pendulum, we can invoke additional symmetry considerations to conclude that $m_0 + m_1 + m_2 \geq 12$, Fig. 3. The integers (0,1,2) in Fig. 3 are the Morse indices of the equilibria of W. For a more detailed discussion of multiple pendula, we refer to [26,27] and references therein.

3.3. A single pendulum

For the single pendulum the above topological argument predicts at least two additional unstable equilibria, Fig. 3. Indeed, the angle x with the upward vertical direction evolves according to

$$\ell\ddot{x} = (g + a(t))\sin x, \quad \langle a\rangle = 0.$$

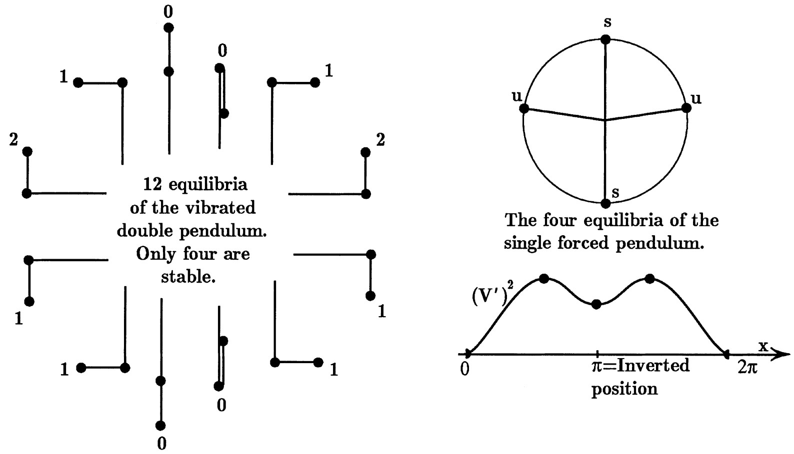

Fig. 3. Topological argument predicts additional equilibria.

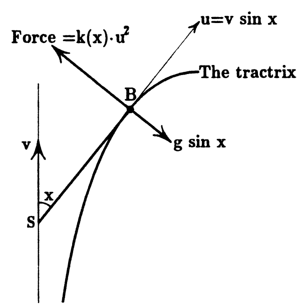

Fig. 4. Stability criterion for the inverted pendulum.

Assume $a(t)$ to be like in Eq. (2); then Theorem 2 gives the averaged truncated equation

$$\ell\ddot{x} = g\sin x - \ell^{-1}\langle v^2\rangle \sin x\cos x, \tag{17}$$

where the error terms have been removed. The effective potential $g\cos x - (1/2\ell)v^2\sin^2 x$ has four critical points, assuming $g\ell < \langle v^2\rangle$, see Fig. 3. For more details on the dynamics near these equilibria we refer to [28].

3.4. Averaging and non-holonomic systems

Here we point out a perhaps surprising connection: the effective force $\ell^{-1}\langle v^2\rangle\sin x\cos x$ in (17) is the force of a certain nonholonomic constraint.

Consider a non-holonomic system in the plane: a segment[4] SB of fixed length ℓ, where the velocity of B is constrained to the line SB, see Fig. 4. As the result of this constraint, if S is guided along a straight line then B will trace a pursuit curve (also referred to as the tractrix)[5]. So far the system is purely kinematic. Let us now endow B with mass $m = 1$; as a result the mass point B, which moves along a tractrix, is subject to the force $F_{\text{centrifugal}} = mu^2/R = ku^2$, where u is the velocity of B, R is the radius of curvature of the tractrix and k is the

[4] S for suspension point, B for bob.

[5] Defined by the property that all the tangent segments have a fixed length: $|SB| = \ell$, where B is an arbitrary point on the curve and S lies on a given straight line.

curvature. A simple calculation shows that $u = v \cos x$, $k = \ell^{-1} \tan x$, so that

$$F_{\text{centrifugal}} = ku^2 = \frac{v^2}{\ell} \sin x \cos x. \tag{18}$$

Remarkably, this coincides with the expression in Eq. (17) obtained by formal averaging. This coincidence is not accidental: it is a manifestation of a general geometric theorem valid for arbitrary potentials on the line, see [25].

One could restate the observation by saying that a *violently vibrated mechanical system has, as its singular limit, a nonholonomic system.*

3.5. A quick derivation of the stability criterion

A closely related idea gives an instant stability criterion for the inverted pendulum, with virtually no calculations. Let us think of the segment SB described above as the pendulum with suspension point S and with the bob B. Thus B is temporarily constrained to the tractrix. We also add gravity to the system, see Fig. 4. As B oscillates along a short arc of a tractrix, it applies the centrifugal force $k\langle u^2 \rangle$ perpendicular to the tractrix. If this force is large enough to dominate the destabilizing component of gravity:

$$k\langle u^2 \rangle > g \sin x, \tag{19}$$

then the bob 'wants' to accelerate towards the top, so that if released from the constraint, it will do so. Eq. (19) is the desired stability criterion!

Since $k = \ell^{-1} x + \mathrm{o}(x)$, Eq. (19) reduces for small x to

$$\langle v^2 \rangle > \ell g, \tag{20}$$

the linearized stability criterion. To obtain even a very special case of Eq. (20), for the piecewise constant a, (cf. [29]) by a standard method takes at least a page of formulae. Of course our method is not rigorous! – but it does give the correct answer (even in the nonlinear case) and its validity is proven by the main theorem.

As a closely related observation we recall the well-known stability condition on the Floquet matrix F of the linear Hamiltonian system cf. Arnold [29]:

$$|\mathrm{tr}\, F| < 2. \tag{21}$$

It is remarkable that the linear algebraic condition (21) has the physical interpretation of the centrifugal force condition (20).

3.6. Curvature of the tractrix in averaged Hill's equations

Curvature of the tractrix shows up in averaging Hill's equation

$$\ddot{x} + q(t, \varepsilon)x = 0, \quad \langle q \rangle = 0 \tag{22}$$

with $q(t, \varepsilon) = \varepsilon^{\alpha} A(t/\varepsilon)$, where $\alpha > -2$ and $A(\tau + 1) = A(\tau)$ and where we take $\langle A \rangle = 0$.

Indeed, the truncated averaged system is, according to Theorem 1

$$\ddot{X} + \kappa \langle v^2 \rangle X = 0, \tag{23}$$

where $\kappa = k'(0)$, $k(x) =$ curvature of the tractrix generated by unit segments ($\ell = 1$).

3.7. Asymptotics of path integrals and curvature

Hill's equation (22) can be written in the Hamiltonian form

$$\dot{\boldsymbol{x}} = A(t,\varepsilon)\boldsymbol{x}, \quad \boldsymbol{x} \in \boldsymbol{R}^2; \quad A = \begin{pmatrix} 0 & 1 \\ -q & 0 \end{pmatrix};$$

the problem of finding the fundamental solution matrix F, i.e. finding the path integral $F(t) = P\exp\left(\int_0^t A(\tau,\varepsilon)\,\mathrm{d}\tau\right)$ can be thought of as finding a product of a continuous family of non-commuting symplectic 2×2 matrices (of the form $\mathrm{e}^{A\mathrm{d}\tau}$). To restate a result from the previous section, the leading term in the asymptotic expression for this product

$$F(t) = \exp\begin{pmatrix} 0 & 1 \\ -\kappa\langle v^2\rangle & 0 \end{pmatrix} + \cdots .$$

4. An informal 'proof' of Theorem 1

In this section, we give a heuristic derivation of the averaged equations. This argument is a minor extension to higher dimension of the argument given by Landau and Lifshitz [8] for the one-dimensional potential systems. We write

$$\boldsymbol{x} = \boldsymbol{X} + \varepsilon^{\beta}\boldsymbol{S}, \quad \beta = 2+\alpha, \tag{24}$$

where $\boldsymbol{X} = \langle \boldsymbol{x}\rangle$, the average over the period $T=\varepsilon$ of a, and $\boldsymbol{S}$ is still to be determined. The scaling factor ε^{β} is designed to absorb the smallness of the amplitude, see Eq. (3).

Substituting Eq. (24) into Eq. (1), we get

$$\ddot{\boldsymbol{X}} + \varepsilon^{\beta}\ddot{\boldsymbol{S}} = a\left(\boldsymbol{f}(\boldsymbol{X}) + \varepsilon^{\beta}\boldsymbol{f}'(\boldsymbol{X})\boldsymbol{S}\right), \tag{25}$$

where the higher order terms $o(a\varepsilon^{\beta})$ have been deleted.

Averaging Eq. (25) over a time interval of length ε gives

$$\ddot{\boldsymbol{X}} = \langle a\rangle\boldsymbol{f}(\boldsymbol{X}) + \varepsilon^{\beta}\boldsymbol{f}'(\boldsymbol{X})\langle a\boldsymbol{S}\rangle, \tag{26}$$

and we must estimate $\langle a\boldsymbol{S}\rangle$. To that end we subtract Eq. (26) from Eq. (25):

$$\varepsilon^{\beta}\ddot{\boldsymbol{S}} = (a-\langle a\rangle)\boldsymbol{f}(\boldsymbol{X}) + \varepsilon^{\beta}\boldsymbol{f}'(\boldsymbol{X})(a\boldsymbol{S} - \langle a\boldsymbol{S}\rangle);$$

keeping only the leading term in the right-hand side, we obtain

$$\varepsilon^{\beta}\ddot{\boldsymbol{S}} = (a-\langle a\rangle)\boldsymbol{f}(\boldsymbol{X})$$

which we use to find $\boldsymbol{S}$. Treating $\boldsymbol{X}$ as the slow variable yields

$$\varepsilon^{\beta}\dot{\boldsymbol{S}} = \int_{t_0}^{t}(a-\langle a\rangle)\,\mathrm{d}s\,\boldsymbol{f}(\boldsymbol{X}) \equiv v(t)\boldsymbol{f}(\boldsymbol{X}) \tag{27}$$

with t_0 chosen so that $\langle v\rangle = 0$. One more integration gives

$$\varepsilon^{\beta}\boldsymbol{S} = \int_{t_1}^{t} v(\tau)\boldsymbol{f}(\boldsymbol{X})\,\mathrm{d}\tau \equiv s(t)\boldsymbol{f}(\boldsymbol{X}),$$

where t_1 is chosen so that $\langle s\rangle = 0$. Having thus found $\boldsymbol{S}$ (to the leading order), we compute $\langle a\boldsymbol{S}\rangle$ using $\langle \boldsymbol{S}\rangle = 0$ in the second step below, integrating by parts in the third step and then using Eq. (27), we obtain

$$\begin{aligned}\langle a\boldsymbol{S}\rangle &= \frac{1}{\varepsilon}\int_t^{t+\varepsilon} a\boldsymbol{S}\,\mathrm{d}\tau = \frac{1}{\varepsilon}\int_t^{t+\varepsilon}(a-\langle a\rangle)\boldsymbol{S}\,\mathrm{d}\tau = -\frac{1}{\varepsilon}\int_t^{t+\varepsilon} v(\tau)\dot{\boldsymbol{S}}\,\mathrm{d}\tau = -\frac{1}{\varepsilon^{\beta}}\frac{1}{\varepsilon}\int_t^{t+\varepsilon} v^2(\tau)\,\mathrm{d}\tau\,\boldsymbol{f}(\boldsymbol{X}) \\ &= -\frac{1}{\varepsilon^{\beta}}\langle v^2\rangle \boldsymbol{f}(\boldsymbol{X}). \end{aligned} \tag{28}$$

Substituting the computed expression into Eq. (26), we obtain the desired averaged equations:

$$\ddot{\boldsymbol{X}} = \langle a\rangle \boldsymbol{f}(\boldsymbol{X}) - \langle v^2\rangle \boldsymbol{f}'(\boldsymbol{X})\boldsymbol{f}(\boldsymbol{X}).$$

5. Proof of Theorem 1: a normal form reduction

This section contains the rigorous proof to compensate for the heuristics of the last section. It should be noted that the main theorem gives an explicit expression of the correction. Consequently, in our normal form reduction, we must keep track of the explicit form of the higher order terms at every step – something not usually done in the normal form reduction. The order of magnitude happens to be $\mathrm{O}(\delta^3)$. One can, of course, keep track of more terms and thus obtain an explicit form up to any order δ^k. It should also be noted that if a and $\boldsymbol{f}$ are analytic, then the system can be brought to an exponentially small perturbation of an autonomous system [30].

5.1. *Rescaling*

We begin by writing our second-order equation as a system in $\boldsymbol{R}^{2n}$:

$$\begin{cases} \dot{x} = y, & x \in \boldsymbol{R}^n \\ \dot{y} = \varepsilon^{\alpha} A(t/\epsilon) f(x), & y \in \boldsymbol{R}^n. \end{cases} \tag{29}$$

Rescaling the time:

$$\tau = \frac{t}{\epsilon}, \tag{30}$$

we obtain, with $' = (\mathrm{d}/\mathrm{d}\tau)$:

$$\begin{cases} x' = \epsilon y \\ y' = \varepsilon^{1+\alpha} A(\tau) f(x). \end{cases} \tag{31}$$

Rescaling y:

$$X = x, \qquad Y = \epsilon^{-\alpha/2}\, y \tag{32}$$

we obtain

$$\begin{cases} X' = \epsilon^{1+\alpha/2}\, Y = \delta Y \\ Y' = \epsilon^{1+\alpha/2}\, A(\tau) f(X) = \delta A(\tau) f(X), \end{cases} \tag{33}$$

where

$$\delta = \epsilon^{1+\alpha/2}, \tag{34}$$

or more compactly,

$$Z' = \delta F(Z,\tau), \qquad F(Z,\tau+1) = F(Z,\tau), \quad Z \in \boldsymbol{R}^{2n}. \tag{35}$$

5.2. O(δ)-*averaging*

We apply the transformation

$$Z = z + \delta h(z, \tau), \tag{36}$$

to Eq. (33), with h to be defined shortly, obtaining

$$\delta h_\tau + (I + \delta h_z) z' = \delta F(z + \delta h, \tau) = \delta F(z, \tau) + \delta^2 F_z h + \boldsymbol{E}_1, \tag{37}$$

where $|\boldsymbol{E}_1| = \delta^3 |F_{zz}(\hat{z}, \tau)| < c\delta^3$. From this, we express

$$z' = \delta(F - h_\tau + \delta\{F, h\}) + \mathrm{O}(\delta^3), \tag{38}$$

where $\{F, h\} = F_z h - h_z F$. First, we specify h so as to make the O(δ)-term time-independent:

$$F(z, \tau) - h_\tau(x, \tau) = \bar{F}(z), \tag{39}$$

(the bar denotes the time-average), i.e. $h(z, \tau) = \int (F(z, \sigma) - \bar{F})\, \mathrm{d}\sigma$; this h is τ-periodic. Moreover, we choose the constant of integration so as to make $\bar{h} = 0$. Explicitly,

$$h(z, \tau) = \int \begin{pmatrix} Y \\ Af \end{pmatrix} - \begin{pmatrix} Y \\ \bar{A} f \end{pmatrix} \mathrm{d}\sigma = \begin{pmatrix} 0 \\ Vf \end{pmatrix}, \tag{40}$$

where [6]

$$V(\tau) = \int A(\sigma) - \bar{A} d\sigma, \qquad \bar{V} = 0. \tag{41}$$

With this h the transformation (36) becomes

$$X = x, \qquad Y = y + \delta V(t) f(x). \tag{42}$$

Applying this transformation to our system (35), we obtain

$$z' = \delta(\bar{F}(z) + \delta G(z, \tau)) + \mathrm{O}(\delta^3), \tag{43}$$

where $G = \{F, h\}$. As it turns out, the τ-average of G vanishes, so that the nature of the O(δ^3)-terms becomes of importance. In the next section, we carry out the averaging while keeping track of the cubic terms.

5.3. O(δ^3)-*averaging*

We apply the transformation $Z = z + \delta h(z, t)$ with h as in Eq. (42) to Eqs. (33):

$$\begin{cases} x' = \delta(y + \delta V f(x)) \\ y' - \delta(\bar{A} - A) f + \delta V f' x' = \delta A f(x), \end{cases} \tag{44}$$

or

$$\begin{cases} x' = \delta(y + \delta V f(x)) \\ y' - \delta \bar{A} f + \delta^2 V f'(y + \delta V f) = 0, \end{cases} \tag{45}$$

[6] This mneumonic notation: $V' = A$, $S' = V$ suggests the analogy with the position, velocity and acceleration.

or

$$\begin{cases} x' = \delta y + \delta^2\, Vf(x) \\ y' = \delta\bar{A}f - \delta^2\, Vf'y - \delta^3\, V^2\, f'f, \end{cases} \tag{46}$$

or equivalently,

$$z' = \delta\bar{F}(z) + \delta^2\, G(z,t) + \delta^3\, K(z,t), \tag{47}$$

where G and K are specified by Eq. (46). In this form, we have retained the information on the $\mathrm{O}(\delta^3)$ terms. To average G, we rename $z = (x, y)$ into $Z = (X, Y)$ in Eq. (47) [7], and use the transformation $Z = z + \delta^2\, h(z, \tau)$ with

$$h = \int (G - \bar{G})\,\mathrm{d}\sigma = \int (V - \bar{V})\,\mathrm{d}\sigma \begin{pmatrix} f \\ -f'y \end{pmatrix} \equiv S \begin{pmatrix} f \\ -f'y \end{pmatrix}, \tag{48}$$

where S is defined by $S(\tau) = \int V(\sigma)\,\mathrm{d}\sigma$, $\bar{S} = 0$. More explicitly, the transformation

$$\begin{cases} X = x + \delta^2\, Sf(x) \\ Y = y - \delta^2\, Sf'(x)y \end{cases} \tag{49}$$

takes Eq. (47) into

$$\begin{cases} x' + \delta^2\, Vf + \delta^2\, Sf'x' = \delta y - \delta^3\, Sf'y + \delta^2\, Vf + \delta^4\, VSf'f + \mathrm{O}(\delta^5) \\ y' - \delta^2\, Vf'y - \delta^2\, Sf''[x', y] - \delta^2\, Sf'y' = \delta\bar{A}f + \delta\bar{A}f'\delta^2\, Sf - \delta^2\, Vf'y - \delta^3\, V^2 f'f + \mathrm{O}(\delta^4), \end{cases} \tag{50}$$

where $f''[\xi, \eta] \equiv f''(\boldsymbol{x})[\xi, \eta]$ denotes the vector-valued second derivative bilinear form in $\xi, \eta \in \boldsymbol{R}^n$ defined by $f''[\xi, \eta] = (\mathrm{d}/\mathrm{d}\lambda)(\mathrm{d}/\mathrm{d}\mu) f(\boldsymbol{x} + \lambda\xi + \mu\eta)|_{\lambda=\mu=0}$. Combining the powers of δ, we obtain

$$\begin{cases} x' = \delta y - \delta^3\, Sf'y - \delta^3\, Sf'y + \delta^4\, VSf'f + \mathrm{O}(\delta^5) \\ y' = \delta\bar{A}f + \delta^3\, S\bar{A}f' + \delta^2\, Sf''[x', y] - \delta^3\, V^2 f'f + \delta^3\, S\bar{A}f'f + \mathrm{O}(\delta^4). \end{cases} \tag{51}$$

This system is implicit in terms of x'; to overcome this problem, we substitute $x' = \delta y + \mathrm{O}(\delta^3)$ into the appropriate term in Eq. (51):

$$\delta^2\, Sf''[x', y] = \delta^3\, Sf''[y, y] + \mathrm{O}(\delta^5), \tag{52}$$

so that Eqs. (51) become

$$\begin{cases} x' = \delta y - 2\delta^3\, Sf'y + \delta^4\, VSf'f + \mathrm{O}(\delta^5) \\ y' = \delta\bar{A}f + \delta^3(2S\bar{A}f'f + Sf''[y, y] - V^2\, f'f) + \mathrm{O}(\delta^4). \end{cases} \tag{53}$$

It should be noted that the quadratic terms have averaged out to zero. One more averaging, this time of the cubic terms (again we rename Z into z, etc.) via $z = Z + \delta^3\, h$, where $h = \int G - \bar{G}$ gives

$$\begin{cases} x' = \delta y + \delta^4\, VSf'f + \mathrm{O}(\delta^5) \\ y' = \delta\bar{A}f - \delta^3\, \langle V^2\rangle f'f + \mathrm{O}(\delta^4). \end{cases} \tag{54}$$

Note that $\mathrm{O}(\delta^4)$-terms are unchanged. With one final averaging, using $\langle VS\rangle = 0$, we obtain

$$\begin{cases} x' = \delta y + \mathrm{O}(\delta^5) \\ y' = \delta\bar{A}f - \delta^3\, \langle V^2\rangle f'f + \mathrm{O}(\delta^4). \end{cases} \tag{55}$$

[7] So that we should have rewritten Eq. (47) with Z instead of z, which we did not do to save space.

5.4. Return to the original variables

Changing from the time τ in Eq. (55) to the original time $t = \varepsilon\tau$ and recalling $\delta = \epsilon^{1+\alpha/2}$, we turn Eq. (55) into

$$\begin{cases} \dot{x} = \epsilon^{\alpha/2}\, y + \mathrm{O}(\epsilon^{4+2.5\alpha}) \\ \dot{y} = \epsilon^{\alpha/2}\, \bar{A} f - \epsilon^{2+(3\alpha/2)} \langle V^2 \rangle f' f + \mathrm{O}(\epsilon^{3+2\alpha}). \end{cases}$$

Recalling that y was rescaled in Eq. (32), we rescale back obtaining

$$\begin{cases} \dot{x} = y + \mathrm{O}(\epsilon^{4+2.5\alpha}) \\ \dot{y} = \epsilon^{\alpha}\, \bar{A} f - \epsilon^{2+2\alpha} \langle V^2 \rangle f' f + \mathrm{O}(\epsilon^{3+2.5\alpha}) \end{cases}$$

From the first equation, we get $\ddot{x} = \dot{y} + \mathrm{O}(\varepsilon^{3+2.5\alpha})$ and thus, using the implicit function theorem

$$\ddot{x} = \epsilon^{\alpha}\, \bar{A} f - \epsilon^{2+2\alpha} \langle V^2 \rangle f' f + \mathrm{O}(\varepsilon^{3+2.5\alpha}).$$

Recalling that $a(t, \varepsilon) = \epsilon^{\alpha} A(t/\varepsilon)$, cf. Eq. (2), we obtain

$$\epsilon^{\alpha}\, \bar{A} = \bar{a}, \qquad \epsilon^{1+\alpha} V(\tau) = v, \tag{56}$$

which gives

$$\ddot{x} = \bar{a} f - \langle v^2 \rangle f' f + \mathrm{O}(\varepsilon^{3+2.5\alpha}), \tag{57}$$

Q.E.D.

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