Lie Brackets and Bulletproof Vests

MATHEMATICAL CURIOSITIES
By Mark Levi

A bullet punctures a wall made of two thin layers A and B of different materials (see Figure 1). Does the exit velocity change if we turn the two-layered "vest" inside out, as in the figure? It turns out that it does, and the difference $v_2 - v_1$ of exit velocities is given by a certain Lie bracket. Let us assume, perhaps unrealistically, that the bullet's deceleration is a given function of its velocity, $dv/dt = a(v)$; for example, one could take $a(v) = -kv^2$. The velocity considered as a function of the position x (rather than the time) satisfies the differential equation $dv/dx = f(v)$ with $f(v) = a(v)/v$. Indeed,

$$\frac{dv}{dx} = \frac{dv}{dt}\frac{dt}{dx} = \frac{a(v)}{v} = f(v). \qquad (1)$$

Similarly, for the second material we have $dv/dx = g(v)$ for some other function g. We thus have two vector fields, f and g, in the one-dimensional *velocity space*; the position x plays the role of time, since we eliminated t in (1).

I claim that the effect of swapping of A and B is given by the Lie bracket:

$$v_2 - v_1 = (g'f - f'g)hk + \ldots = [g,f]hk + \ldots, \qquad (2)$$

for small h and k. Here, $\ldots = o(h^2 + k^2)$.

The Lie bracket $[g,f]$ can be the difference between life and death (although this difference is small).

Proof

Let F^x denote the flow of the vector field f. In other words, v_0 becomes $F^x v_0$ after the bullet travels distance x in material A; the flow G^x is defined similarly for material B. Then

$$v_1 = G^k F^h v_0, \quad v_2 = F^h G^k v_0,$$

and

$$v_2 - v_1 = F^h G^k v_0 - G^k F^h v_0,$$

This difference is given by the right side of (2), as verified by a Taylor expansion of $G^x v_0$ and $F^x v_0$ to second order; for example,

$$F^x v_0 = v_0 + fx + \frac{1}{2} f' f x^2 + \ldots.$$

I omit further details.

An example

If $f(v) = -v^\alpha$, $g(v) = -v^\beta$, then we get $[f,g] = (\beta - \alpha)v^{\alpha+\beta-1}$, and so only for $\alpha = \beta$ is the "vest" reversible.

I conclude with a trivial but curious observation, worth mentioning in an ordinary differential equation or mechanics course or a faculty lounge: if $dv/dt = f(v)$, then $K = \frac{v^2}{2}$ satisfies $dK/dx = f(v)$. Thus, $dv/dt = dK/dx$. *Velocity changes with respect to time exactly like kinetic energy changes with respect to distance.*

As a special case of this remark, *the kinetic energy decays exponentially with the distance* for the commonly-assumed quadratic drag $f = -cv^2$. Indeed, we have $dK/dx = f(v) = -2cK$.

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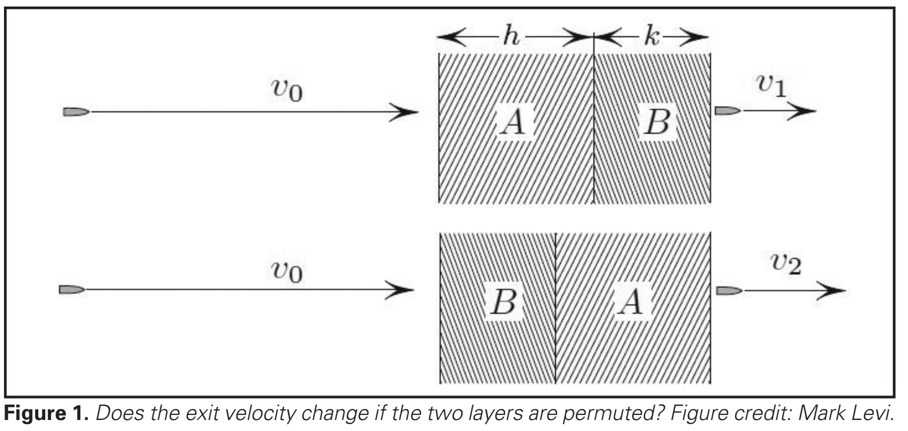

Figure 1. *Does the exit velocity change if the two layers are permuted? Figure credit: Mark Levi.*