Some Light Geometry

Of the two geometrical curiosities below, the first involves zero work and the second (almost) zero words.

1. This physical "proof" of the Pythagorean theorem involves no work – mechanical work, that is. Figure 1(a) shows a right triangle, its hypotenuse held vertical. We take a point mass of the same weight c as the length of the hypotenuse,[1] so that c plays a double role of the length and of the weight. Lifting the mass along AB requires the same work as dragging it up the slippery ramps AC and CB:

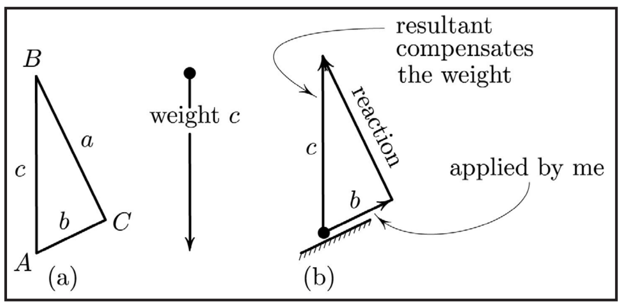

Figure 1. *Forces felt by the mass as one drags it up AC (with no acceleration) add up to zero. Thus, the resultant of the reaction of the ramp and the supplied force compensates the gravitational force. The triangle formed by these two forces is congruent to ΔABC, and thus the force one must apply is b. In short, ΔABC plays two roles, a geometrical one and a force one.*

MATHEMATICAL CURIOSITIES
By Mark Levi

$$W_{AB} = W_{AC} + W_{CB}. \quad (1)$$

Indeed, had the left-hand side been, say, smaller, we could have cycled the weight along the closed path $ABCA$, extracting more energy on the way down than we spent on the way up – a functioning perpetual motion machine.

Now (1) gives the theorem, since $W_{AB} = c \cdot c = c^2$, $W_{AC} = b \cdot b = b^2$, and $W_{CB} = a^2$, as explained by Figure 1 for W_{AC}. The Pythagorean theorem is thus one consequence of the constant vector field's conservativeness.

2. A wordless proof of the formula $\sin(\beta - \alpha) = \sin\beta\cos\alpha - \sin\alpha\cos\beta$, referring to Figure 2, expresses the same area in two different ways:

$$A = 1 \cdot 1 \cdot \sin(\beta - \alpha) =$$
$$\begin{vmatrix} \cos\alpha & \cos\beta \\ \sin\alpha & \sin\beta \end{vmatrix} =$$
$$\sin\beta\cos\alpha - \sin\alpha\cos\beta.$$

[1] in some chosen units

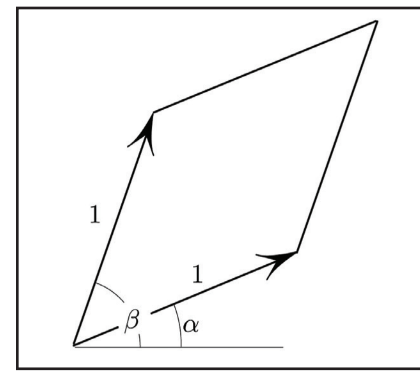

Figure 2. *A is the area of the parallelogram.*

Mark Levi (levi@math.psu.edu) is a professor of mathematics at the Pennsylvania State University.