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Schrödinger's equation and "bike tracks" — A connection

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The purpose of this note is to demonstrate an equivalence between two classes of objects: the stationary Schrödinger equation on the one hand, and the "bicycle tracks" on the other. We begin with the description of the latter.

A (very) idealized model of a bicycle, shown in Fig. 1, is a segment RF of constant length which is allowed to move in the plane as follows: the path of the "front" F is prescribed, while the velocity of the "rear" R is constrained to the line RF: the "rear wheel" does not sideslip. The bike length $|RF| = 1$ is fixed throughout this note, without the loss of generality. If $(X(t), Y(t))$ is a parametric representation of the motion of F, then the angle θ between the unit length bike RF and the x-axis in the plane satisfies the differential equation

$$\dot{\theta} = \dot{Y}\cos\theta - \dot{X}\sin\theta, \tag{1}$$

expressing the fact that infinitesimal displacement of R is aligned with the direction $e^{i\theta}$ of the segment. Eq. (1) can be rewritten as

$$\dot{\theta} = -v\sin(\theta - \varphi), \tag{2}$$

where, referring to Fig. 2, v and φ are defined by $ve^{i\varphi} = \dot{X} + i\dot{Y}$ and by the condition that φ is continuous. Some examples of tracks are given in Fig. 3. Yet another equivalent way to write this equation (see [1], equation (4)) is to introduce the steering angle $\alpha = \varphi - \theta$ (see Fig. 2), to observe that $\dot{\varphi} = v\kappa$, where κ is the curvature of the front track, and to substitute the last two relations into (2), with the resulting ODE for the steering angle α:

$$\dot{\alpha} = -v\sin\alpha + v\kappa. \tag{3}$$

A very brief history. The idealized "bike" of Fig. 1 has been studied since the second half of 19th century (see [2] and references therein), and up to the present time [2,3]. It was observed that the "bike" arises as an asymptotic limit of a system describing a particle in a rapidly oscillating potential; it is interesting that the nonholonomic "bike" is a singular limit of a holonomic system (the details can be found in [4], and in [5]).

Stationary Schrödinger's equation

$$\ddot{x} + p(t)x = 0 \tag{4}$$

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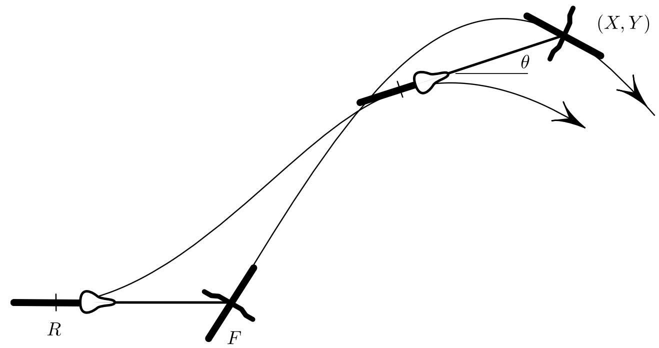

Fig. 1. An idealized bike. In this example *F* travels along an arc of a sine curve.

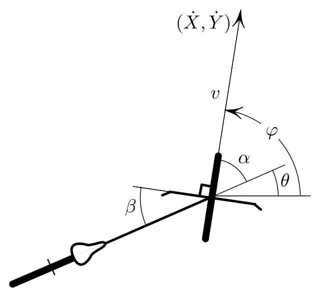

Fig. 2. Different forms of the bicycle equation.

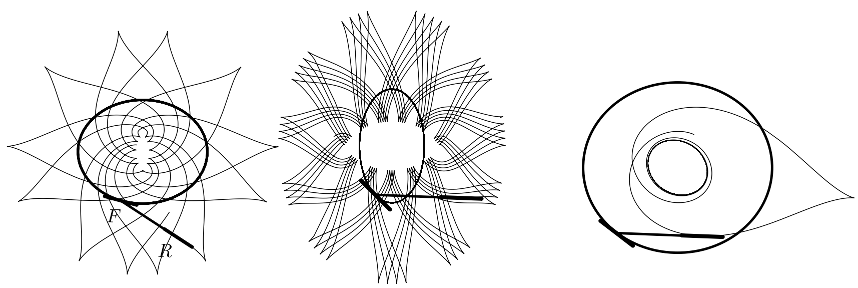

Fig. 3. The path traced out by the rear wheel as the front wheel repeatedly traverses a closed curve.

is a classical object of mathematical physics, arising in many settings in mathematics, physics and engineering. This system has been studied for nearly two centuries. Known also as Hill's equation, it comes up in studying the spectrum of hydrogen atom, in celestial mechanics [6], in particle accelerators [7], in forced vibrations, in wave propagation, and in many more problems. Hill's operator deforms isospectrally when its potential evolves under the Korteweg–de Vries (KdV) equation, thus providing an explanation of complete integrability of the latter [8–10]. The 1989 Nobel Prize in physics was awarded to W. Paul for his invention of an electromagnetic trap, now called the Paul trap, used to suspend charged particles. The mathematical substance of Paul's discovery amounts to an observation on Hill's equation, as explained in Paul's Nobel lecture [11]. Incidentally, [12] contains a geometrical explanation, as an alternative to Paul's analytical one, of why the trap works. Stability of the famous Kapitsa pendulum [13,14] is also explained by the properties of Hill's equation (Stephenson gave an experimental demonstration of stability of the so-called Kapitsa pendulum in 1908 [15], about half a century before Kapitsa's paper). The long history of Hill's equation is reflected in the rich body of classical literature of the 18th and 19th centuries on the eigenfunctions of special second order equations (polynomials of Lagrange, Laguerre, Chebyshev, Airy's function, etc.), to the more recent work on inverse scattering and on geometry of "Arnold tongues" [16–21,10,22–24].

1. The main result

To any potential p in Hill's equation (4) we will assign a front wheel path $(X(t), Y(t))$ (via (5)–(6)). With such an assignment, the two systems: the bike equation (1) (or (2), or (3)) on the one hand and Hill's equation (4) on the other

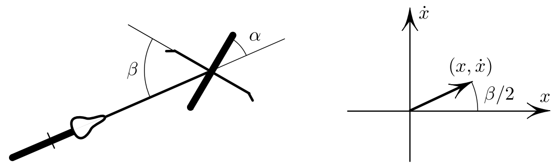

Fig. 4. The "handlebar angle" β evolves exactly as the doubled phase angle $2\arg(x+i\dot{x})$ of (4), provided that the front wheel's path is generated by the potential $p(t)$ via (5)–(6).

become equivalent in the sense explained by Fig. 4. Roughly speaking, for any given Hill's equation, the projectivization of its phase plane carries a bicycle flow.

Theorem 1. *Let a Schrödinger potential $p(t)$ in (4) be given. We associate with p the front wheel path $(X(t), Y(t))$ as follows: defining*

$$\varphi(t) = t + \int_0^t p(s)\,ds + \frac{\pi}{2}, \tag{5}$$

we set

$$\begin{cases} X(t) = \int_0^t (1-p(\tau))\cos\varphi(\tau)\,d\tau \\ Y(t) = \int_0^t (1-p(\tau))\sin\varphi(\tau)\,d\tau. \end{cases} \tag{6}$$

If the potential p and the path (X, Y) are thus related, then the two problems: the Schrödinger equation (4) and the bike problem (1) are equivalent in the sense that

$$\theta = 2\arg(x+i\dot{x}) + \varphi - \frac{\pi}{2}, \tag{7}$$

where φ is given by (5).[1] *Equivalently, (7) can be written more geometrically in terms of the handlebar angle $\beta = \pi/2 - \alpha$ shown in Fig. 4, namely*

$$\beta = 2\arg(x+i\dot{x}). \tag{8}$$

Eq. (8) gives a direct bicycle interpretation to the phase angle for Hill's equation.

Fig. 5 shows front paths corresponding to various potentials.

Remark 1. The potential p is related to the angular velocity $\omega = \dot{\varphi}$ of the handlebar via $p = \omega - 1$. Indeed, $\omega \overset{def}{=} \frac{d}{dt}\arg(\dot{X} + i\dot{Y}) \overset{(6)}{=} \dot{\varphi} \overset{(5)}{=} 1 + p$.

Remark 2. The handlebar angular velocity ω is, up to the sign, the 2D curl of the vector field of the associated Schrödinger system (14). This is seen from the second version of the proof of the theorem (see (20)).

Remark 3. If follows from (5) and (6) that the potential p is related to the curvature κ of the front track via

$$\kappa = \frac{1+p}{1-p}.$$

Remark 4. The reason for the appearance of factor 2 in (7) and (8) is explained by the lemma in the beginning of Section 2.

A reformulation of the main result.

The track (6) can be thought of as the path of a particle subject to a strange magnetic-like force defined in the next paragraph.

A pseudo-magnetic force. Let $v = v(t)$ be a given function of time, and consider a point mass $m = 1$ moving in the plane with speed v and subject to normal acceleration due to the following magnetic-like force:

$$\mathbf{F} = \mathbf{a}_\perp = i(2 - v)\mathbf{v} \tag{9}$$

[1] More precisely, if (7) holds for $t = 0$, then it holds for all t.

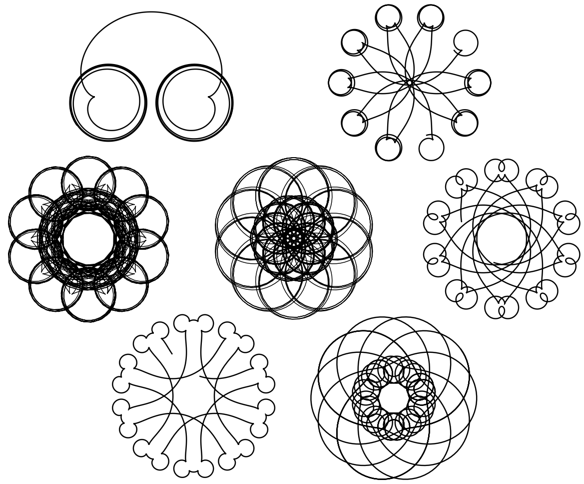

Fig. 5. Front wheel paths representing different potentials. The top left corresponds to a soliton of the KdV. The remaining ones are generated by various trigonometric potentials. Alternatively, these paths are trajectories of the particle subject to the "magnetic" force (10) with different prescribed speeds $v = v(t)$.

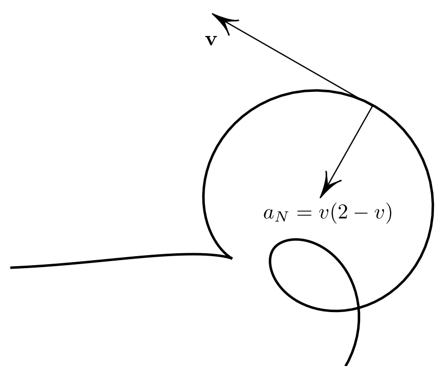

Fig. 6. Trajectory of a particle subject to force (9) with v prescribed. At the cusp v changes sign.

acting normal to the velocity $\mathbf{v}$. Note that the tangential velocity v is prescribed (one can imagine a tangential force acting on the particle in addition to the normal force (9)), and that the normal acceleration is slaved to v. We allow v to change sign, so that $v = \pm|\mathbf{v}|$; if v changes sign, the particle reverses the direction of motion, as illustrated in Fig. 6.

The main result can now be reformulated as follows.

Theorem 2. *Consider the Schrödinger equation (4) with potential $p(t)$. Define*

$$v(t) = 1 - p(t), \tag{10}$$

and let $(X(t), Y(t))$ be a path of the "magnetic" particle defined in the preceding paragraph. Then the Schrödinger equation (4) and the bike problem (1) are equivalent in the sense of Theorem 1.

2. Proofs

The proof of Theorem 1 relies on the following simple fact about linear systems.

Lemma 1. *For a linear system*

$$\begin{cases} \dot{u} = au + bv \\ \dot{v} = cu + dv \end{cases} \tag{11}$$

(with the possibly time-dependent coefficients), the angle $\chi = \arg(u + iv)$ of any nontrivial solution satisfies the first order ODE

$$\frac{d}{dt} 2\chi = A + B\cos 2\chi + C\sin 2\chi, \tag{12}$$

where

$$A = c - b, \quad B = c + b, \quad C = d - a.$$

Proof. We have

$$\dot{\chi} = \frac{d}{dt}\arg(u+iv) = \frac{d}{dt}\operatorname{Im}\ln(u+iv) = \operatorname{Im}\frac{\dot{u}+i\dot{v}}{u+iv}.$$

Substituting the derivatives from (11) leads to the quadratic trigonometric polynomial on the right-hand side:

$$\dot{\chi} = c\cos^2\chi + (d-a)\cos\chi\sin\chi - b\sin^2\chi;$$

expressing the right-hand side in terms $\cos 2\chi$ and $\sin 2\chi$ results in (12). ◇

Proof of Theorem 1. We recall the "bicycle equation" for the steering angle α (equation (4) in [1]):

$$\dot{\alpha} = -v\sin\alpha + v\kappa \tag{13}$$

(to derive this equation, one substitutes $\varphi - \theta = \alpha$ (Fig. 4) and $\dot{\varphi} = v\kappa$ into (2)).

On the other hand, consider the Schrödinger system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -p(t)x; \end{cases} \tag{14}$$

according to the above lemma, the double angle $2\arg(x+i\dot{x}) = \beta$ of any solution satisfies the ODE

$$\dot{\beta} = -1 - p + (1-p)\cos\beta; \tag{15}$$

this form is similar to that of (13), apart from the difference between sine and cosine. We eliminate this difference by introducing the complementary angle

$$\beta_c = \frac{\pi}{2} - \beta,$$

thus turning (15) into

$$\dot{\beta}_c = -(1-p)\sin\beta_c + 1 + p. \tag{16}$$

For the bicycle equation (13) to coincide with this one, we must choose the path $(X(t), Y(t))$ so that

$$v = 1 - p, \qquad kv = 1 + p; \tag{17}$$

but these relations hold precisely if the path (X, Y) is constructed via (6)–(5) (and conversely, (17) implies (6)–(5) modulo the Euclidean motions of the path). We showed that

$$\alpha = \beta_c \tag{18}$$

in the sense that the two satisfy the same ODE; and since $\alpha = \varphi - \theta$ (Fig. 2) and $\beta_c = \frac{\pi}{2} - 2\arg(x+i\dot{x})$ (by the definition), (18) gives (7). ◇

An alternative proof of the theorem

In the above proof we projectivized the Schrödinger's equation and established the equivalence of this projectivization with the bicycle equation. The proof could proceed instead on the level of linear systems: we could "anti-projectivize" the bicycle equation, by realizing it as a linear ODE in $\mathbb{R}^2$ and then transforming this ODE into the Schrödinger system (14). Here is an outline of this alternative approach.

Starting with the bicycle equation in the form (1), we wish to lift it to a linear system. Reading Lemma 1 backwards, we conclude that (1) lifts to the linear system

$$\dot{w} = \frac{1}{2}\begin{pmatrix} \dot{X} & \dot{Y} \\ \dot{Y} & -\dot{X} \end{pmatrix} w \tag{19}$$

in the sense that $\arg w$ satisfies (1). Now both this and Schrödinger's system (14) are Hamiltonian, i.e. both have traceless matrices; however, the 2D curl of the vector field (19) is zero (since the matrix is symmetric), while that of (14) is

$$\frac{\partial}{\partial x}(-px) - \frac{\partial}{\partial y}y = -p - 1. \tag{20}$$

That is, the average angular velocity of radial rays in the Schrödinger (x, y)-plane is $-(p+1)/2$. This suggests that the Schrödinger vectors (x, y) subjected to counter-rotation with angular velocity $(p+1)/2$ are moving according to a zero-curl vector field, i.e. the one given by a symmetric matrix. This matrix is also traceless (because the flow is divergence-free and the rotation is area-preserving), and hence has the form

$$\begin{pmatrix} r & s \\ s & -r \end{pmatrix}. \tag{21}$$

In more detail, let

$$\psi(t) = \int_0^t (1 + p(\tau))d\tau \tag{22}$$

be the cumulative angle, with the corresponding rotation matrix

$$R = R(t) = \begin{pmatrix} \cos\psi/2 & -\sin\psi/2 \\ \sin\psi/2 & \cos\psi/2 \end{pmatrix}.$$

Let us now write the Schrödinger system (14) in matrix form $\dot{z} = Pz$; the solutions subjected to counter-rotation, i.e. vectors $w = Rz$, satisfy

$$\dot{w} = (R^{-1}PR - R^{-1}\dot{R})w.$$

A simple computation confirms the previously mentioned expectation that the matrix of this system has the form (21) with

$$r = -\frac{1-p}{2}\sin\psi, \qquad s = \frac{1-p}{2}\cos\psi.$$

This matrix (21) of the transformed Schrödinger system will coincide with the bicycle matrix in (19) if X, Y satisfy

$$\dot{X} = -(1-p)\sin\psi, \qquad \dot{Y} = (1-p)\cos\psi.$$

By setting $\psi = \varphi - \pi/2$ we obtain (5) and (6). And $w = Rz$ translates to $\arg w = \arg z + \psi$, so that

$$\theta \equiv 2\arg w = 2\arg z + \psi = 2\arg z + \varphi - \frac{\pi}{2},$$

proving (7). ◇

Proof of Theorem 2. Consider the motion $(X(t), Y(t))$ given by (6). The velocity of this motion is

$$\begin{cases} \dot{X} = (1 - p(t))\cos\varphi \\ \dot{Y} = (1 - p(t))\sin\varphi. \end{cases} \tag{23}$$

The speed $v = 1 - p$ is in the direction φ if $1 - p > 0$ and in the opposite direction if $1 - p < 0$. The angular velocity of this motion is

$$\omega = \dot{\varphi} = 1 + p,$$

and thus the normal acceleration

$$a_\perp = \omega v.$$

But

$$\omega = 1 + p \overset{(10)}{=} 1 + (1 - v) = 2 - v,$$

so that $a_\perp = v(2 - v)$. ◇

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References

[1] S. Tabachnikov, Tire track geometry: variations on a theme, Israel J. Math. 151 (2006) 1–28.
[2] R.L. Foote, Geometry of the prytz planimeter, Rep. Math. Phys. 42 (1–2) (1998) 249–271.
[3] M. Levi, S. Tabachnikov, On bicycle tire tracks geometry, menzin's conjecture, and oscillation of unicycle tracks, Exp. Math. 18 2 (2009) 173–186.
[4] M. Levi, W. Weckesser, Stabilization of the inverted linearized pendulum by vibration, SIAM Rev. 37 (2) (1995) 219–223.
[5] M. Levi, Geometry and physics of averaging with applications, Physica D 132 (1999) 150–164.

[6] C.L. Siegel, J.K. Moser, Lectures on Celestial Mechanics, in: Grundlehren der mathematischen Wissenschaften, Vol. 87, Springer, 1971.
[7] H. Wiedemann, Particle Accelerator Physics, Springer-Verlag, Berlin Heidelberg, 2007.
[8] C.S. Gardner, R.M. Miura, M.D. Kruskal, Korteweg–de Vries equation and generalizations. ii. Existence of conservation laws and constants of motion, J. Math. Phys. 9 (1968) 1204–1209.
[9] P.D. Lax, Integrals of nonlinear equations of evolution and solitary waves, Comm. Pure Appl. Math. 21 (1968) 67–490.
[10] S.P. Novikov, A periodic problem for the korteweg-de vries equation. i, Funct. Anal. Appl. 8 (3) (1974) 236–246.
[11] W. Paul, Electromagnetic traps for charged and neutral particles, Revs. Mod. Phys. 62 (1990) 531–540.
[12] M. Levi, Stability of the inverted pendulum – a topological explanation, SIAM Rev. 30 (1988) 639–644.
[13] V.I. Arnold, Ordinary Differential Equations, Springer Verlag, 1992.
[14] P.L. Kaptisa, Dynamical stability of a pendulum when its point of suspension vibrates, in: Collected Papers by P.L. Kapitsa, Vol. II, Pergamon Press, London, 1965, pp. 714–725.
[15] A. Stephenson, On a new type of dynamical stability, Manch. Mem. 52 (1908) 1–10.
[16] V.I. Arnold, Remarks on the perturbation theory for problems of mathieu type, Russian Math. Surveys 38 (4) (1983) 215–233.
[17] H.W. Broer, C. Simo, Resonance tongues in hill's equations: a geometric approach, J. Differential Equations 166 (2) (2000) 290–327.
[18] H.W. Broer, M. Levi, Geometrical aspects of stability theory for hill's equations, Arch. Ration. Mech. Anal. 131 (1995) 225–240.
[19] I.M. Gelfand, B.M. Levitan, On the determination of a differential equation from its spectral function, Amer. Math. Soc. Transl. 1 (2) (1955) 253–304.
[20] D.M. Levy, J.B. Keller, Instability intervals of hill's equation, Comm. Pure Appl. Math. 16 (1963) 469–476.
[21] V.A. Marchenko, Sturm–Liouville Operators and Applications, Birkhäuser Verlag, Basel, 1986.
[22] B. van der Pol, M.J.O. Strutt, On the stability of the solutions of mathieu's equation, in: The London, Edinburgh and Dublin Phil. Mag. 7th series, vol. 5, 1928.
[23] M.I. Weinstein, J.B. Keller, Hill's equation with a large potential, SIAM J. Appl. Math. 45 (1985) 200–214.
[24] M.I. Weinstein, J.B. Keller, Asymptotic behavior of stability regions for hill's equation, SIAM J. Appl. Math. 47 (1987).