Reflecting on Reflections

MATHEMATICAL CURIOSITIES

By Mark Levi

I. When looking at a shiny ball bearing in my outstretched hand, I see the reflection of my head. This tiny image seems to lie somewhere inside the sphere. Where exactly? More precisely, what is the limiting position of the point *I* (*Image*) in Figure 1, as $MA \parallel NB$ approaches *NB*? The answer turns out to be *the midpoint of the radius OB*.

Figure 1 explains why. We have

$$\angle 1 \overset{A}{=} \angle 2 \overset{B}{=} \angle 3 \overset{C}{=} \angle 4,$$

where *A* holds because the incoming rays are parallel, *B* is the law of reflection, and *C* holds because the two angles are vertical. Summarizing, $\angle 1 = \angle 4 \overset{def}{=} \theta$, making the triangle *OAI* equilateral and implying that

$$OI = \frac{r}{2\cos\theta} \to \frac{r}{2} \quad \text{for} \quad \theta \to 0,$$

as claimed. V.I. Arnold's engaging book [1] contains a derivation of this fact, although it takes slightly more than a page of calculation.

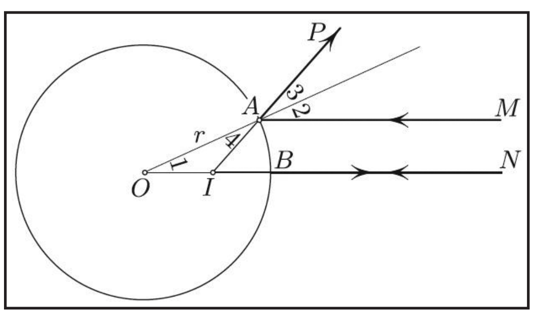

Figure 1. *The reflected ray AP seems to emanate from the point I. And I approaches the midpoint of the radius OB as the ray MA approaches the ray NB.*

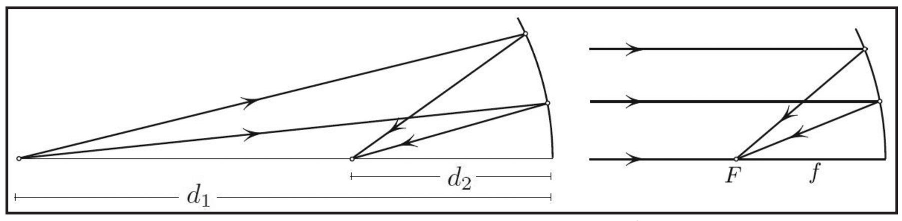

Figure 2. *Illustration of the mirror formula. If* $d_1 = \infty$, *then* $d_2 = f$.

II. An even shorter derivation of the image location results from the mirror formula, which states that the source-to-mirror distance d_1 and the image-to-mirror distance d_2 satisfy

$$\frac{1}{d_1} + \frac{1}{d_2} = \frac{1}{f}, \qquad (1)$$

where *f* is the focal length of the mirror (see Figure 2). But setting $d_1 = r$ (the radius of curvature of the mirror at *B*) yields $d_2 = r$ (see Figure 3), since the rays emanating from the center of curvature collect back at the center (infinitesimally speaking, i.e., replacing the reflector by its osculating circle). In short, (1) yields

$$\frac{1}{r} + \frac{1}{r} = \frac{1}{f},$$

of $f = r/2$, as claimed.

III. Figure 4 shows how the mirror formula comes out of the reflection law $\theta_1 = \theta_2$. We have $\theta_1 = \angle 3 - \angle 1$, where $\angle 3 = ks + o(s)$ (k being the curvature at *B*), and $\angle 1 = s/d_1 + o(s)$. With the similar expression for θ_2, the reflection law amounts to

$$ks - \frac{s}{d_1} = \frac{s}{d_2} - ks + o(s).$$

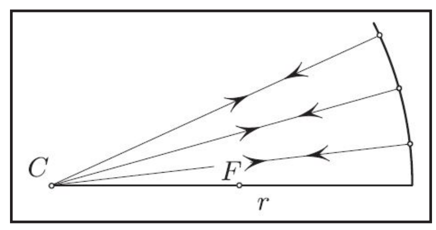

Figure 3. *If* $d_1 = r$ *(the radius of curvature), then* $d_2 = r$.

Dividing by s and taking the limit for $s \to 0$ gives[1]

$$\frac{1}{d_1} + \frac{1}{d_2} = 2k.$$

Substituting $d_1 = \infty$ and $d_2 = f$, we conclude that $2k = 1/f$. This proves the mirror formula (1), and reproduces the fact that the focal distance is half the radius of curvature, $f = 1/2k = r/2$. Incidentally, we proved this for any smooth curve, not necessarily a circle.

[1] Retaining the names d_1, d_2 for the limiting values of the distances, in a mild abuse of notation.

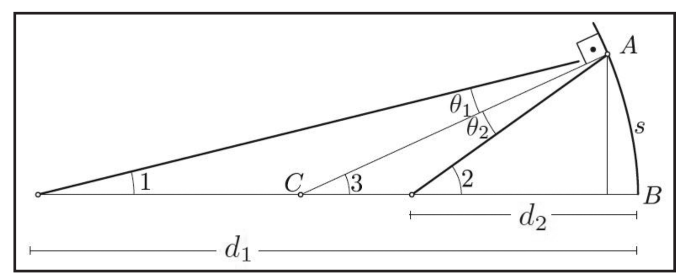

Figure 4. *Proof of the mirror formula.*

The figures in this article were provided by the author.

References

[1] Arnold, V.I. (2014). *Mathematical Understanding of Nature: Essays on Amazing Physical Phenomena and Their Understanding by Mathematicians*. Providence, RI: American Mathematical Society.

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