

# Poincaré Lecture

## Counterintuitive effects in mechanics

Stanford University  
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“Therefore geometry is ... nothing but that part of universal mechanics which accurately proposes and demonstrates the art of measuring”

Newton, Principia

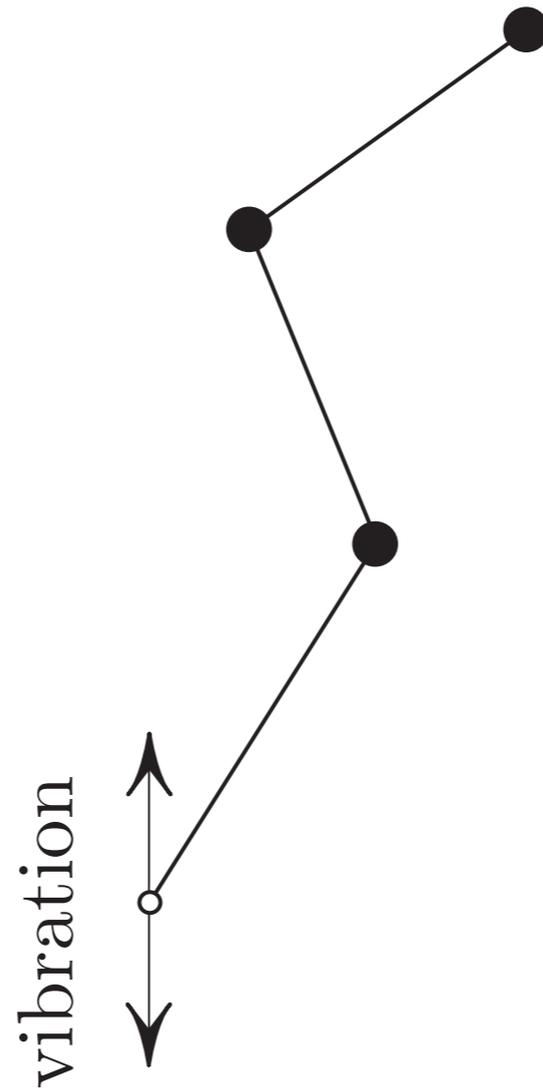
“...what is perfectly accurate is called geometrical ; what is less so, is called mechanical.”

Kapitsa:

It is natural that no mechanical system has been the object of so much attention and comprehensive theoretical study as various forms of the pendulum motion. One would think that in the course of 300 years since the time of Galilei this problem must have been settled and that, if anything of the sort remained to be studied, then it should have the character of improving results found earlier. But, obviously, insufficient attention was paid to the type of pendulum motion to which the present article is devoted, and so one very peculiar and interesting form of the pendulum oscillations remained almost completely uninvestigated.

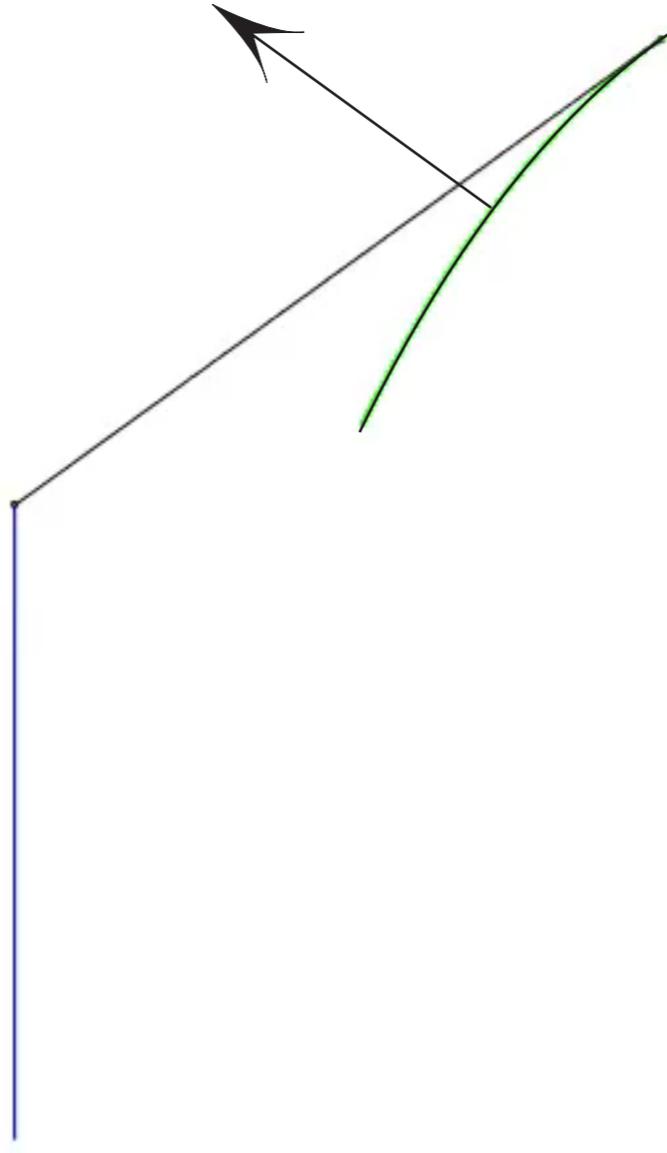
# The Stephenson-Kapitsa effect





Even multiple pendula have been stabilized in experiments (Acheson and Mullin, Nature, 1993).

# The Stephenson-Kapitsa effect



## A geometrical explanation

The standard approach: write the ODE

$$l\ddot{\theta} + (-g + a(t, \omega)) \sin \theta = 0 \quad \text{where } a = \dot{v}; \quad (1)$$

use the normal form calculation to arrive at the averaged *autonomous* equation (deleting higher order terms):

$$l\ddot{\varphi} - g \sin \varphi + \frac{v^2}{l} \sin \varphi \cos \varphi = 0. \quad (2)$$

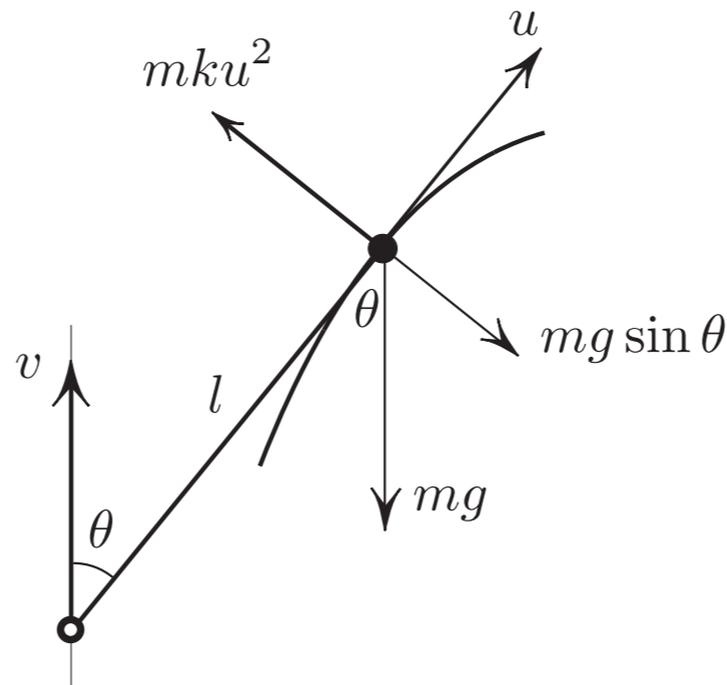
**A geometrical idea, instead:** Imagine constraining the velocity  $u$  to be aligned with the rod. There arises a centrifugal force straining against the constraint:

$$F_{\text{centrif}} = mku^2$$

With  $k = l^{-1} \tan \theta$  and  $u = v \cos \theta$  (the rod's length is fixed) this

$$F_{\text{centrif}} = ku^2 = \frac{v^2}{l} \sin \theta \cos \theta.$$

This is exactly the term in (2)!



*Solution.* The equation of motion can be written in the form  $\ddot{x} = (\omega^2 \pm d^2)x$  (the sign changes after time  $\tau$ ), where  $\omega^2 = g/l$  and  $d^2 = c/l$ . If the oscillation of the suspension is fast enough, then  $d^2 > \omega^2$  ( $d^2 = 8a/l\tau^2$ ).

As in the previous problem,  $A = A_2 A_1$ , where

$$A_1 = \begin{pmatrix} \operatorname{ch} k\tau & \frac{1}{k} \operatorname{sh} k\tau \\ k \operatorname{sh} k\tau & \operatorname{ch} k\tau \end{pmatrix} \quad A_2 = \begin{pmatrix} \cos \Omega\tau & \frac{1}{\Omega} \sin \Omega\tau \\ -\Omega \sin \Omega\tau & \cos \Omega\tau \end{pmatrix}$$

$$k^2 = d^2 + \omega^2, \quad \Omega^2 = d^2 - \omega^2.$$

The stability condition  $|\operatorname{tr} A| < 2$  therefore has the form

$$(7) \quad \left| 2 \operatorname{ch} k\tau \cos \Omega\tau + \left( \frac{k}{\Omega} - \frac{\Omega}{k} \right) \operatorname{sh} k\tau \sin \Omega\tau \right| < 2$$

We will show that this condition is fulfilled for sufficiently fast oscillations of the point of suspension, i.e., when  $c \gg g$ . We introduce the dimensionless variables  $\varepsilon, \mu$ :

$$\frac{a}{l} = \varepsilon^2 \ll 1 \quad \frac{g}{c} = \mu^2 \ll 1.$$

## 5: Oscillations

Then

$$k\tau = 2\sqrt{2}\varepsilon\sqrt{1+\mu^2} \quad \Omega\tau = 2\sqrt{2}\varepsilon\sqrt{1-\mu^2}$$

$$\frac{k}{\Omega} - \frac{\Omega}{k} = \sqrt{\frac{1+\mu^2}{1-\mu^2}} - \sqrt{\frac{1-\mu^2}{1+\mu^2}} = 2\mu^2 + \mathcal{O}(\mu^4).$$

Therefore, for small  $\varepsilon$  and  $\mu$  we have the following expansion with error  $\mathcal{O}(\varepsilon^4 + \mu^4)$ :

$$\operatorname{ch} k\tau = 1 + 4\varepsilon^2(1 + \mu^2) + \frac{8}{3}\varepsilon^4 + \dots \quad \cos \Omega\tau = 1 - 4\varepsilon^2(1 - \mu^2) + \frac{8}{3}\varepsilon^4 + \dots$$

$$\left( \frac{k}{\Omega} - \frac{\Omega}{k} \right) \operatorname{sh} k\tau \sin \Omega\tau = 16\varepsilon^2\mu^2 + \dots$$

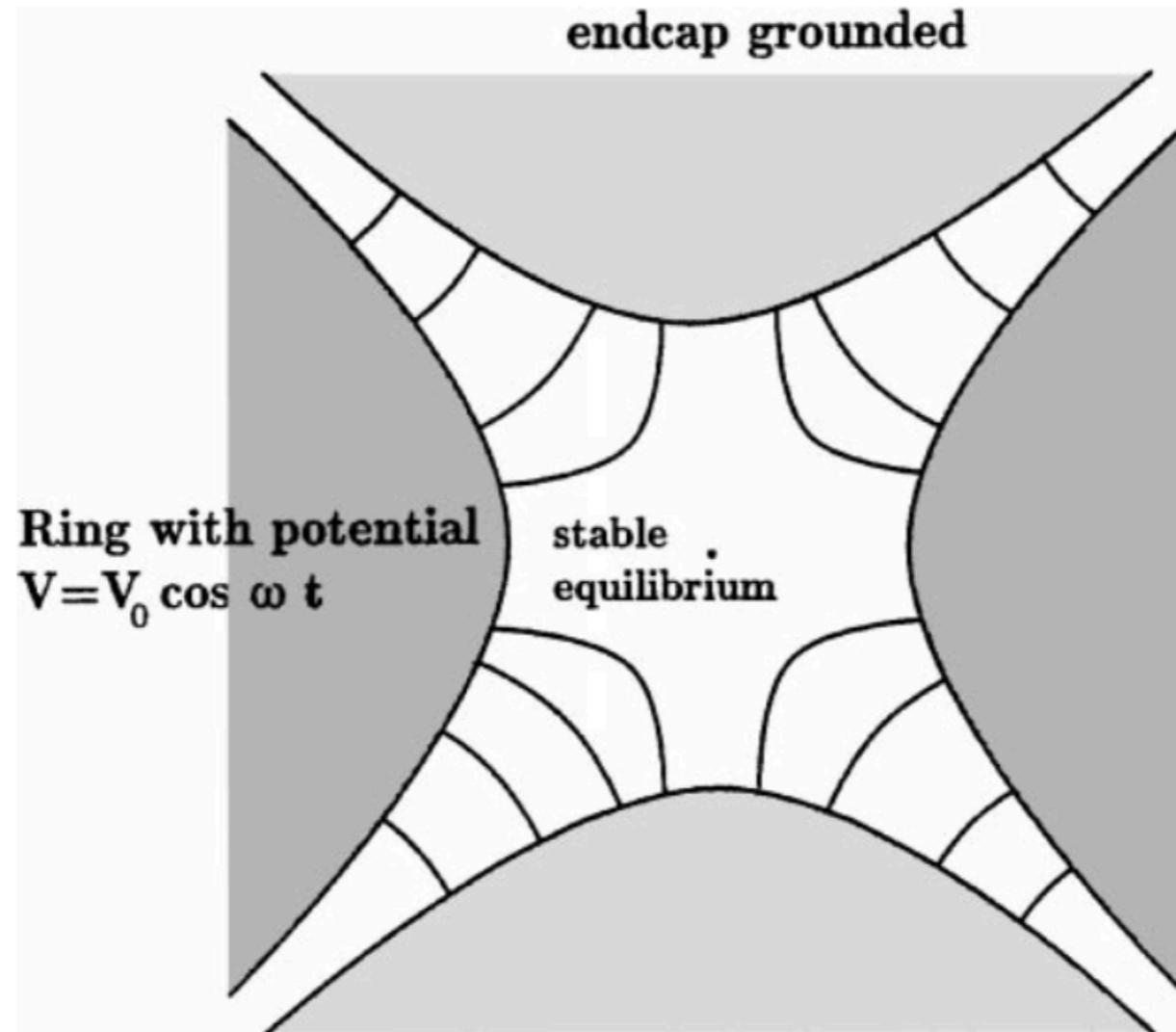
so the stability condition (7) takes the form

$$2(1 - 16\varepsilon^4 + \frac{16}{3}\varepsilon^4 + 8\varepsilon^2\mu^2 + \dots) + 16\varepsilon^2\mu^2 < 2,$$

i.e., disregarding the small higher-order terms,  $\frac{4}{3}16\varepsilon^4 \geq 32\mu^2\varepsilon^2$  or  $\mu \leq \varepsilon\sqrt{2/3}$ , or  $g/c \leq 2a/3l$ . This condition can be rewritten as

$$N \geq \sqrt{\frac{3}{64}} \omega \frac{l}{a} \approx 0.22 \omega \frac{l}{a},$$

# Paul trap



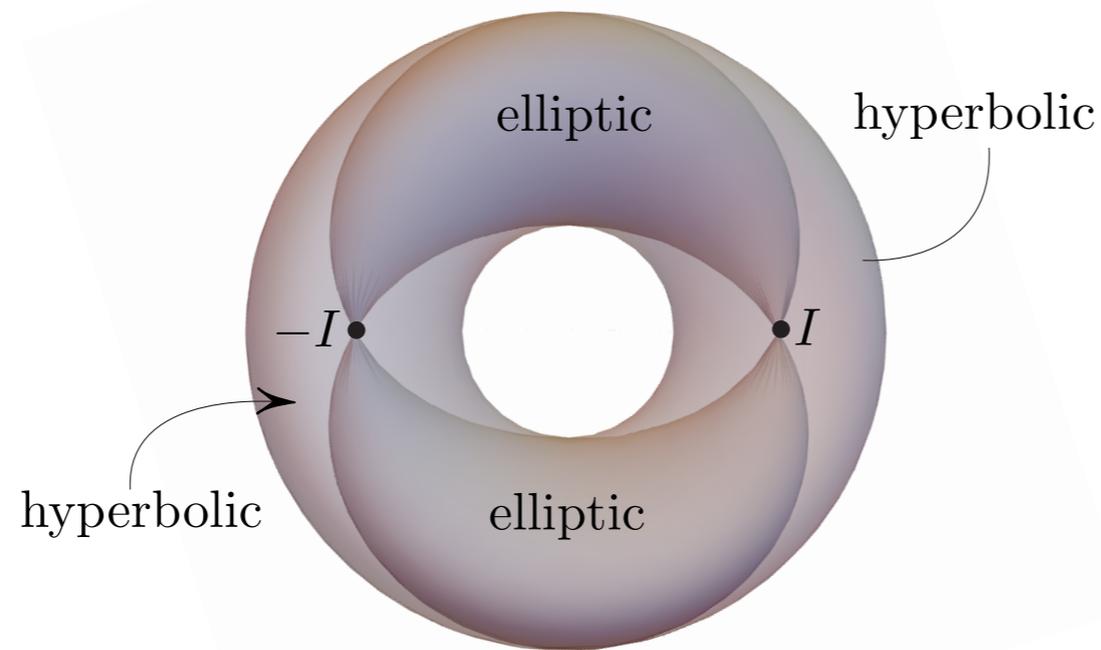
## A topological explanation of stability

**Theorem** If the downward acceleration of the pivot exceeds free fall then there exists an infinite sequence of disjoint open  $L$ -intervals clustering at  $L = 0$  for which the pendulum is stable.

For  $L \in$  the gaps, strong stability is lost.

## A quick background on $SL(2, \mathbb{R})$

Topologically,  $SL(2, \mathbb{R})$  is the interior of the open solid torus, i.e.  $\mathbb{R}^2 \times \mathbb{S}^1$ .



**Proof:** Every  $M \in SL(2, \mathbb{R})$  has a unique polar decomposition:  $M = PR$ , where  $P = P^T > 0$  and  $R \in SO(2)$ .

The set of such  $P$  is topologically  $\mathbb{R}^2$ ; the set of rotations is the circle  $\mathbb{S}^1$ .

**Moreover,** elliptic regions “obstruct” the torus.

**Proof:**  $PR_{\pi/2}$  is elliptic for any  $P = P^T > 0$ .

## A topological explanation of stability-cont'd

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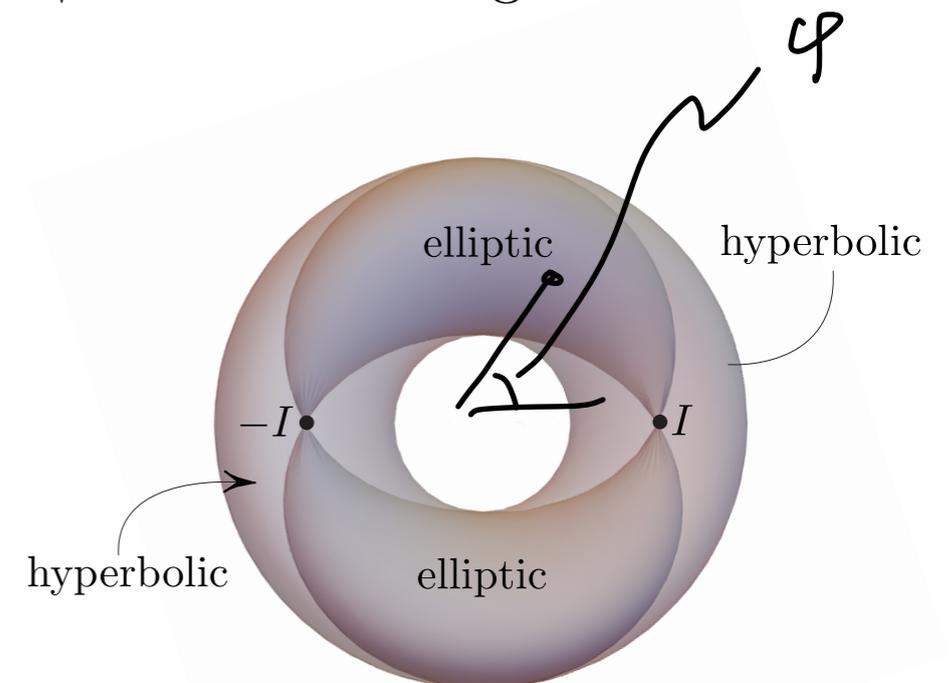
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**Idea of the proof.**

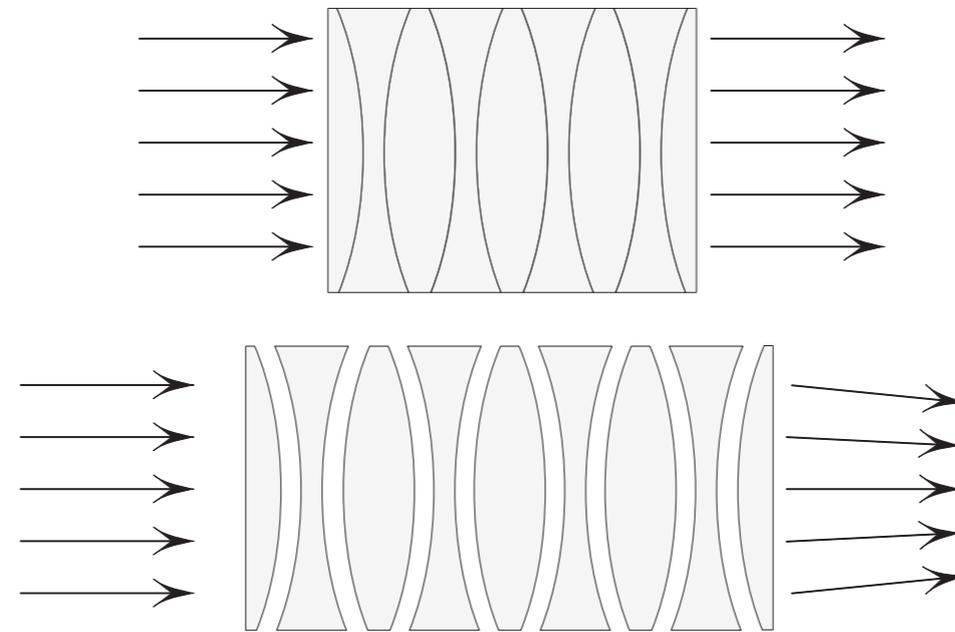
$$\mathbf{x}(T) = F(T)\mathbf{x}(0) = P(T)Rot(T)\mathbf{x}(0)$$

where  $P = P^T > 0$ . As the pendulum's length  $L \downarrow 0$ ,  $\arg \mathbf{x}(T) \rightarrow \infty$ . Hence the angle of  $Rot(T) \rightarrow \infty$  (Keller-Maslov index). Thus  $X(T)$  enters and leaves elliptic regions infinitely many times as  $L \downarrow 0$  since these regions obstruct the torus.  $\diamond$

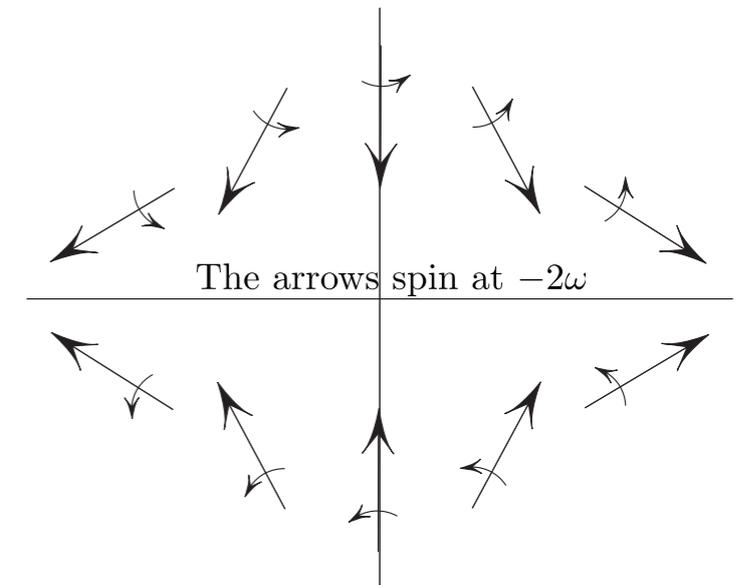
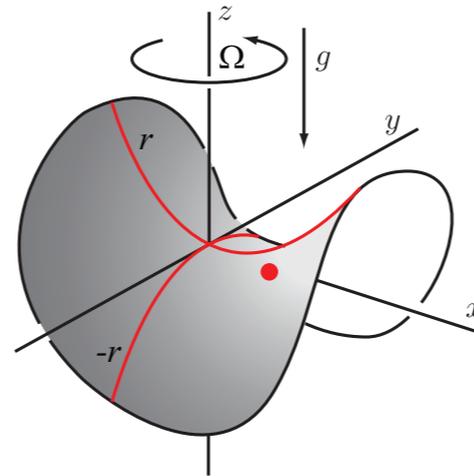


## A related optical effect

In the alternating focusing and defocusing of “equal strengths” the focusing wins – just like with the alternating stability and instability stability wins.



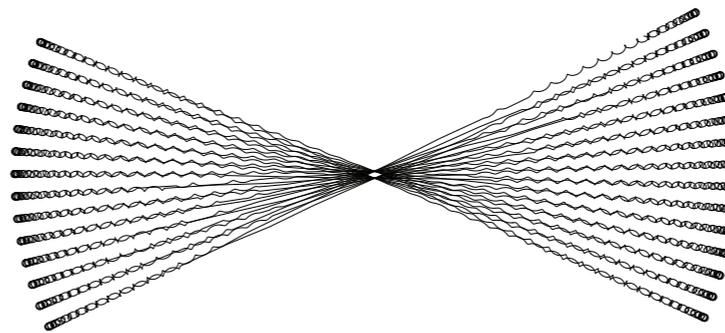
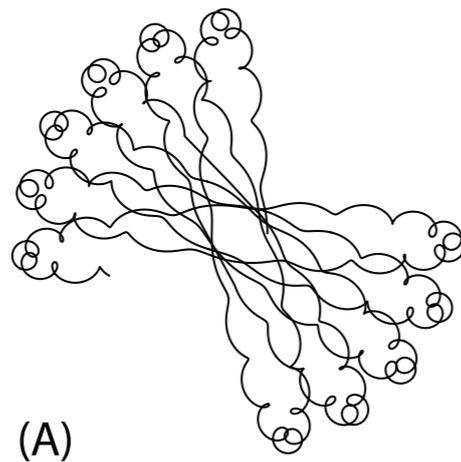
# Stabilization by rotation



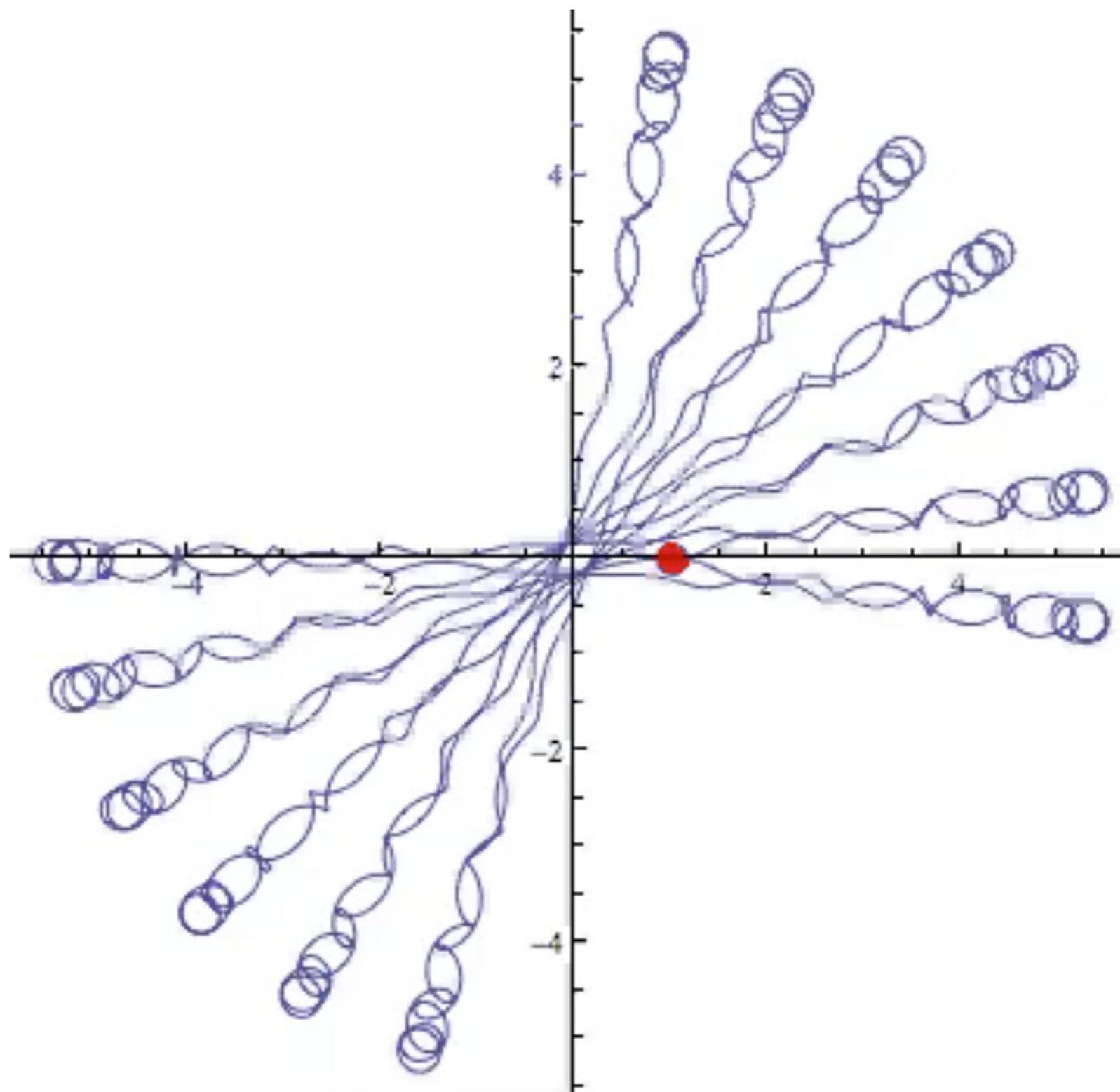
First studied by Brouwer (of the Bohl-Brouwer fixed point theorem).

The saddle, e.g.  $U(x, y) = x^2 - y^2$  is rotated with angular velocity  $\omega$ :  
 $U^t = U \circ R(-\omega t)$ .

Surprising stability for large  $\omega$  (a heuristic explanation).



Animation:



## Ponderomotive magnetism

Take any time-periodic potential  $U(x, \tau) = U(x, \tau + 1)$ ,  $U : \mathbb{R}^n \times \mathbb{R} \rightarrow \mathbb{R}$ .

particle in a rapidly oscillating potential field obeys

$$\ddot{x} = -U_x(x, t/\varepsilon)$$

## Ponderomotive magnetism

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particle in a rapidly oscillating potential field obeys

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**Theorem** [Cox-L] There exists a transformation  $X = x + \varepsilon^2 \phi(x, \dot{x}, t, \varepsilon)$  such that the “guiding center”  $X$  behaves as a particle in a time-independent potential subject to a Lorentz-like magnetic-like force:

$$\ddot{X} = -\overline{U}'(X) - \varepsilon^2 W'(X) + \underbrace{\varepsilon^3 B(X) \dot{X}}_{\text{“magnetic”}} + O(\varepsilon^4),$$

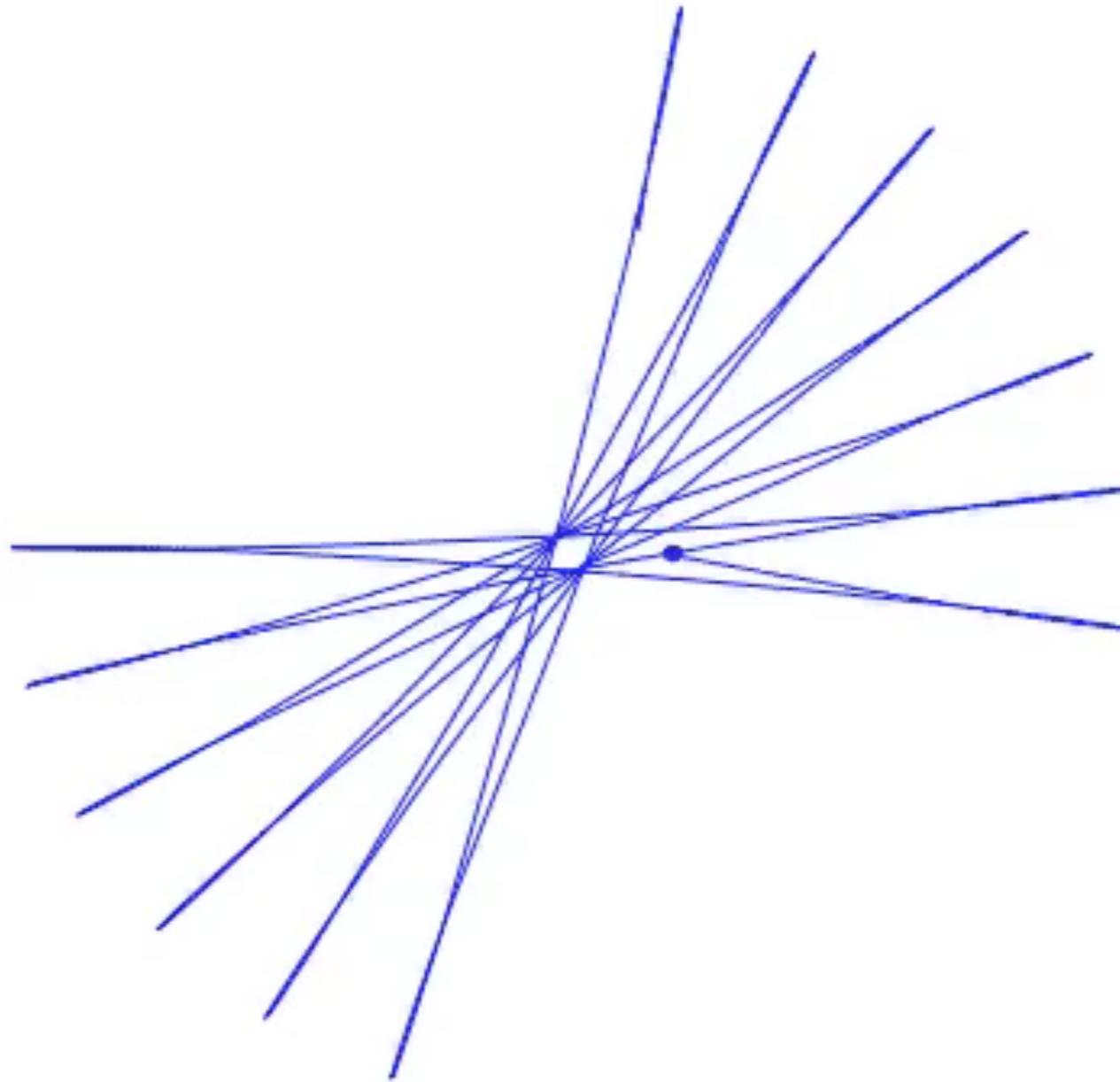
where

' is the gradient w.r.t.  $X$ ,  $\overline{U}$  is the time-average of  $U$ ,

$W : \mathbb{R}^n \rightarrow \mathbb{R}$  is given in terms of  $U$  and

$B = B(X)$  is an antisymmetric matrix given in terms of  $U$ .

True solution  $x$  (red) follows the blue "guiding center"  $X$ :

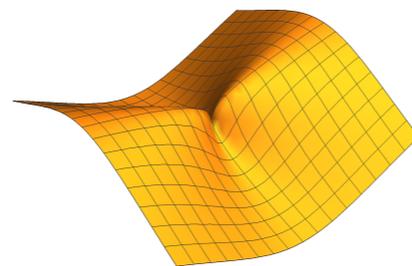


## Two examples

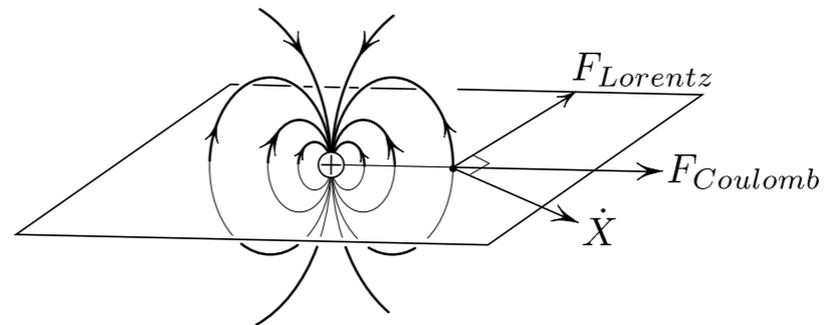
**Example 1** :  $U(x, t/\varepsilon) = \sqrt{r} \cos 2(\theta - t/\varepsilon)$ . Guiding center obeys

$$\ddot{X} = \underbrace{\varepsilon^2 \alpha \frac{X}{|X|^3}}_{\text{Coulomb}} - \underbrace{\varepsilon^3 \beta \frac{J\dot{X}}{|X|^3}}_{\text{Lorentz}}$$

( $\alpha = 17/64$ ,  $\beta = -13/32$ ).



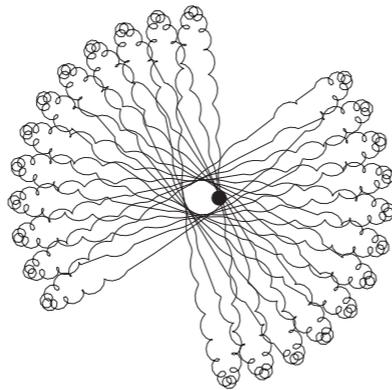
Graph of  $U$



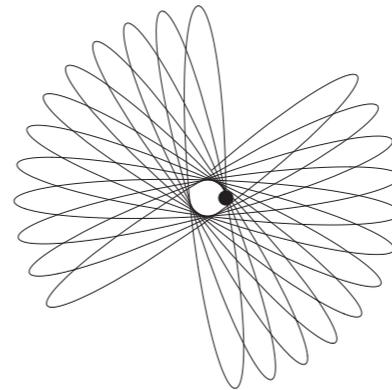
## Two examples

**Example 2:**  $U(x, t)$  is the saddle  $x^2 - y^2$  rotating with the angular velocity  $1/\varepsilon$ . Then

$$\ddot{X} = \underbrace{-\varepsilon^2 X}_{\text{attracting}} + \underbrace{\varepsilon^3 J \dot{X}}_{\text{"magnetic"}} + O(\varepsilon^4).$$

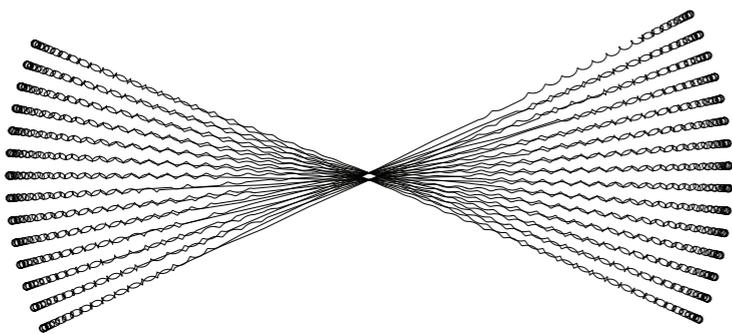
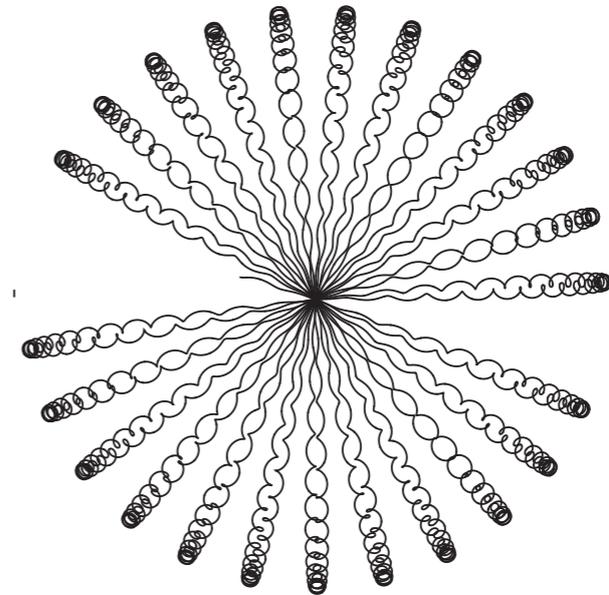
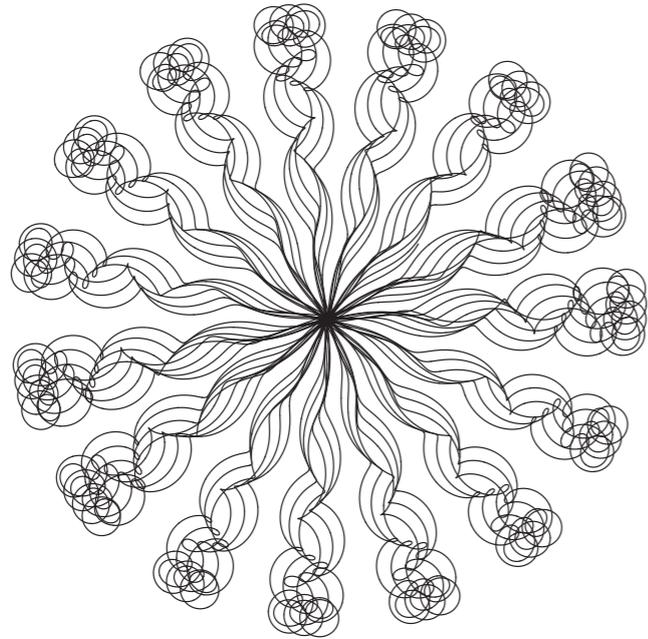


Trajectory of  $x$



Trajectory of  $X$

Large  $\omega$  means tighter/slower precession



Not including the last term  
leads to wrong qualitative picture,  
missing the precession

## Remarks

**1.** For  $n = 3$  the “magnetic force”  $\varepsilon^3 B(X)\dot{X} = q\mathbf{v} \times \mathbf{B}$ , where  $q = \varepsilon^3$ ,  $\mathbf{v} = \dot{X}$  and  $\mathbf{B} \in \mathbb{R}^3$  is defined by the skew-symmetric  $B$ .

**2.** Adiabatic invariant: the energy

$$E(x, \dot{x}) = \frac{\dot{X}^2}{2} + \bar{U}(X) + \varepsilon^2 W(X),$$

(where  $X$  is associated with  $x$  per theorem) satisfies  $\dot{E} = \mathcal{O}(\varepsilon^4)$ . For  $\alpha \in (0, 4)$   $E$  changes by  $O(\varepsilon^{4-\alpha})$  over the time  $0 < t < \varepsilon^\alpha$ .

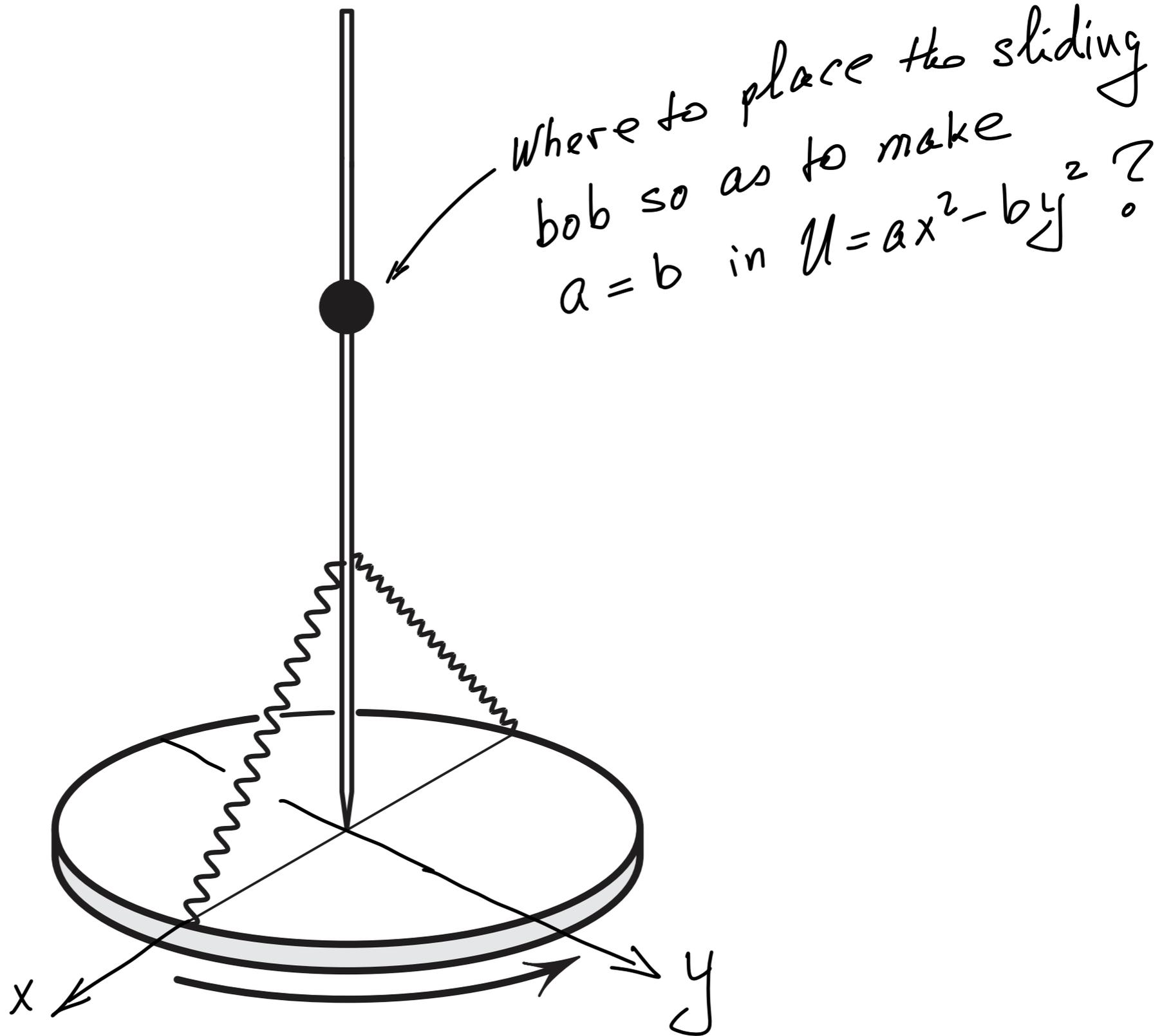
**3.** Another adiabatic invariant if  $n = 2$ :

$$I(x, \dot{x}) = X \wedge \dot{X} + \int_0^{|X|} r\beta(r) dr$$

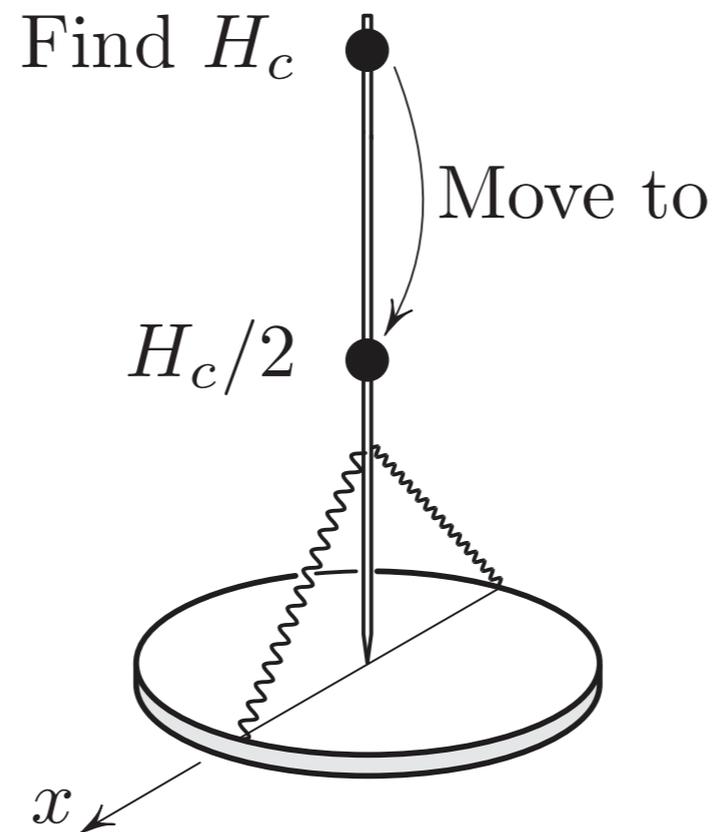
(where  $X$  is associated with  $x$  per theorem) where  $\beta = \text{curl } b$  (the 2D scalar curl) can be thought of as magnetic field perpendicular to the plane.

So  $I(x, \dot{x})$  is a combination of the angular momentum and the flux of magnetic field through the disk  $r < |X|$ .

An experimental demo:

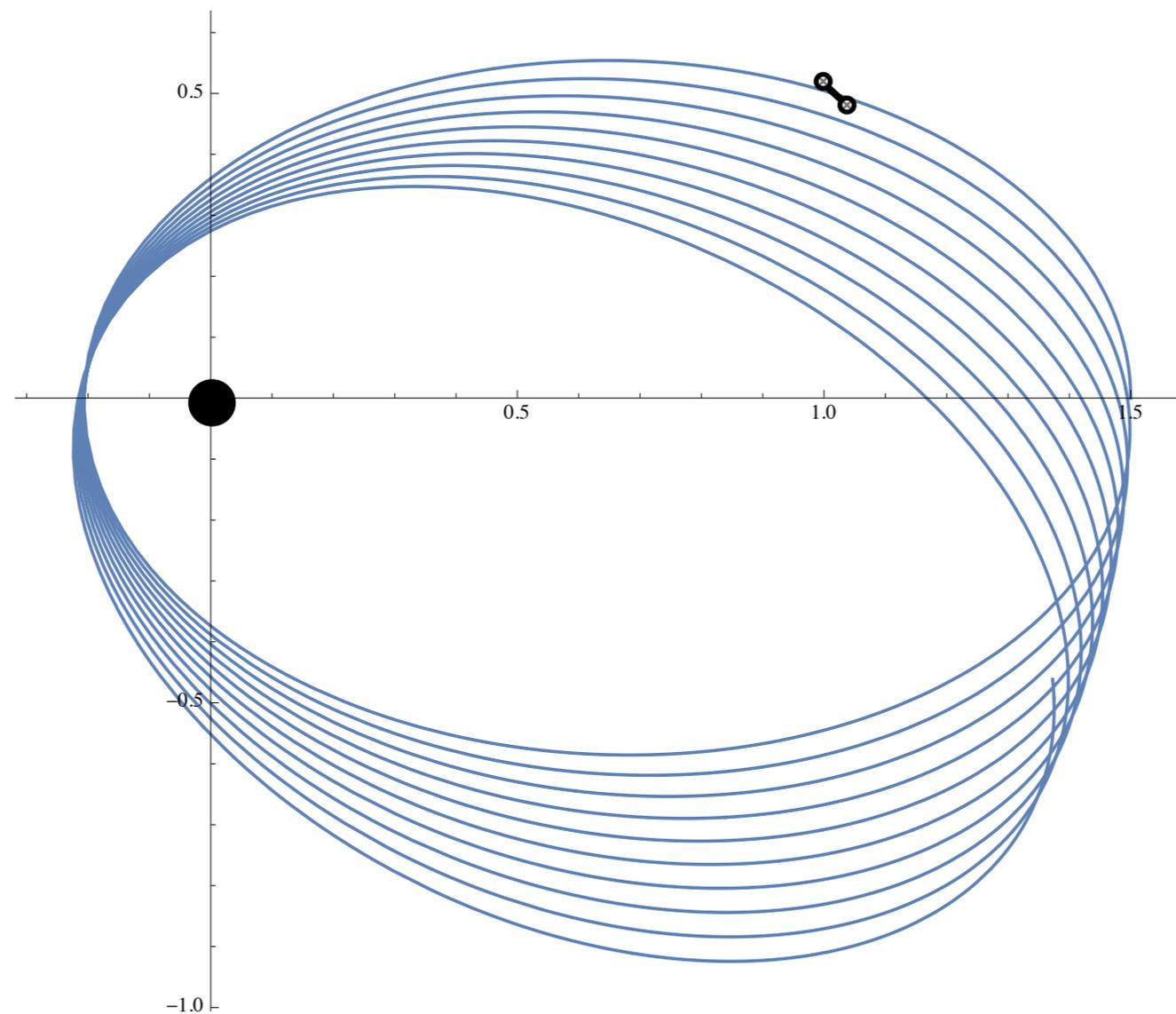


Making  $a = b$  in the saddle  $U_0 = ax^2 - by^2$ .

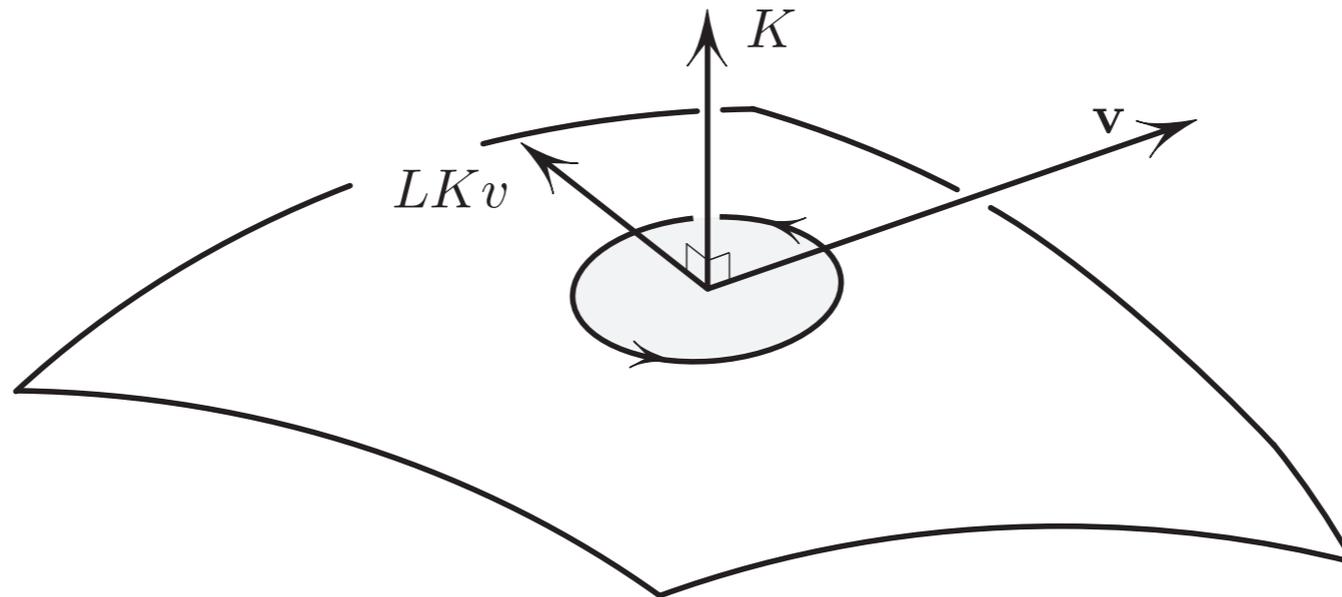


Magnetic-like effects arise in other contexts. For example, a tight binary, viewed as a single point mass, is subject to a magnetic-like force (our recent work with Graham Cox).

This “magnetic” force causes the Kepler ellipses to precess:



## Gaussian curvature and the gyroscopes



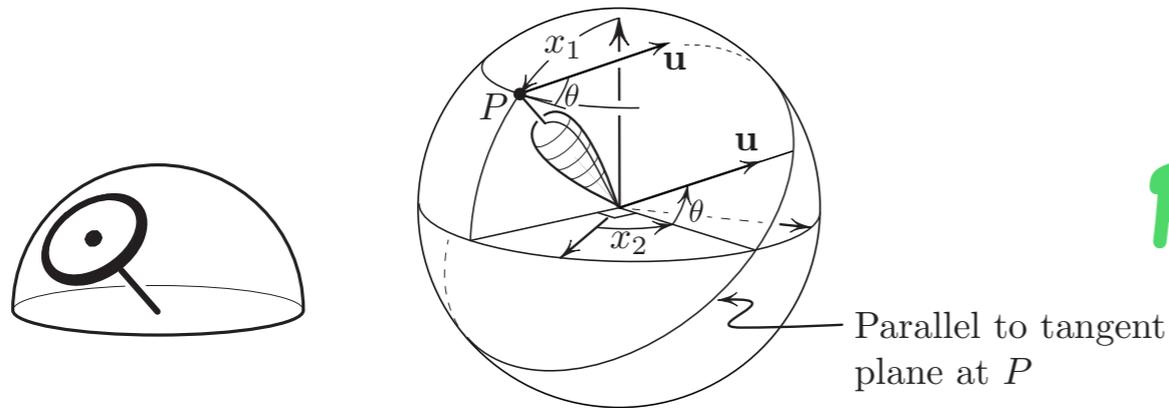
**Theorem [Cox-L].** Spinning disk constrained to be tangent to the surface and sliding with speed  $v$  is subject to the gyroscopic Lorentz-like force orthogonal to the velocity:

$$F = \underbrace{L}_q \underbrace{K}_B \underbrace{v}_v,$$

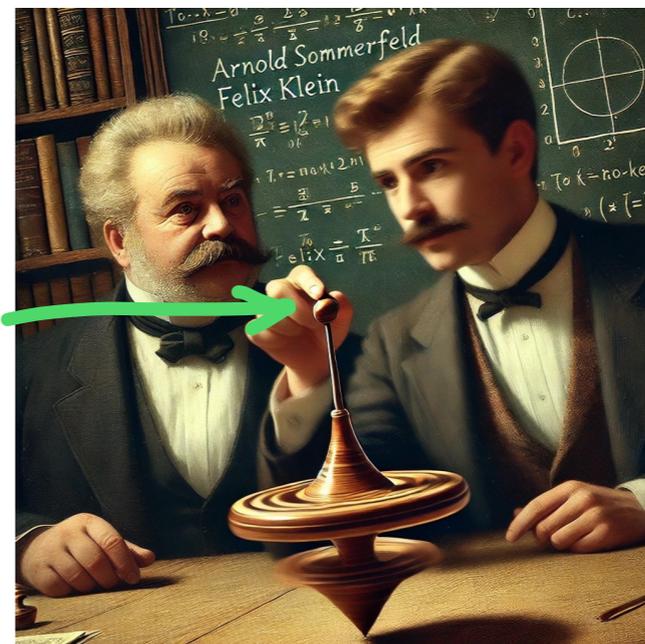
where  $L = I\omega$  is the angular momentum playing the role of charge and the Gaussian curvature  $K$  playing the role of magnetic field orthogonal to the surface.

For such trajectories

$$k_{\text{geodesic}} = \lambda K, \quad \lambda = \text{const.}$$



$P$



### An application: the Lagrange top:

Any Lagrange top can be replaced by an equivalent spinning disk tangent to a sphere. The axial moment of inertia of the disk  $I_a = I_3$  is the same as the top's. The radius of the sphere and the mass  $m$  of the disk are given by

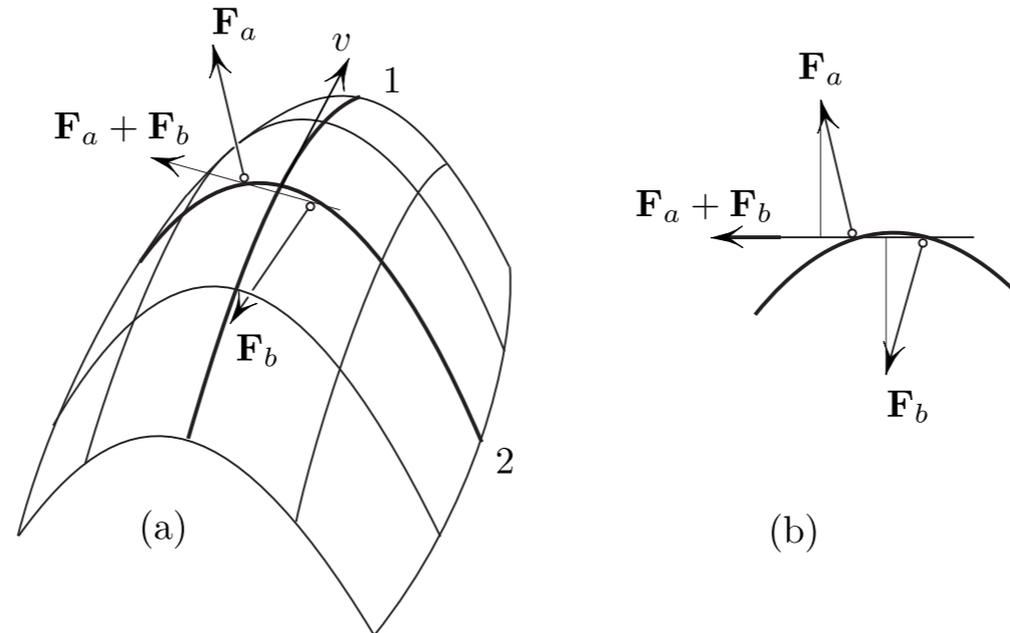
$$R = \frac{I_1}{M\ell}, \quad m = \frac{M^2\ell^2}{I_1}$$

where  $M$  is the mass of the top,  $\ell$  is the distance of the top's center of mass to the ball joint,  $I_1 = I_2$  are the other two moments of inertia of the top.

This observation simplifies the standard treatment of the Lagrange top, e.g. in Goldstein or Arnold.

The Gaussian curvature of the sphere:  $R^{-2}$  is hidden in all the standard treatments of the Lagrange top.

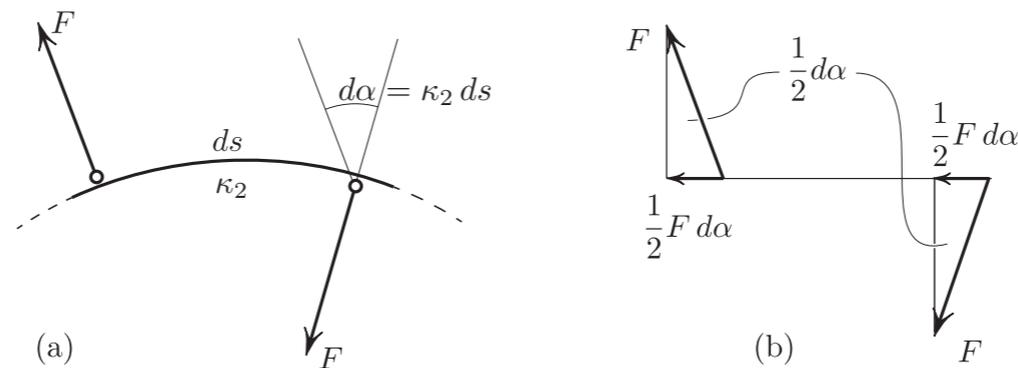
## A heuristic explanation:



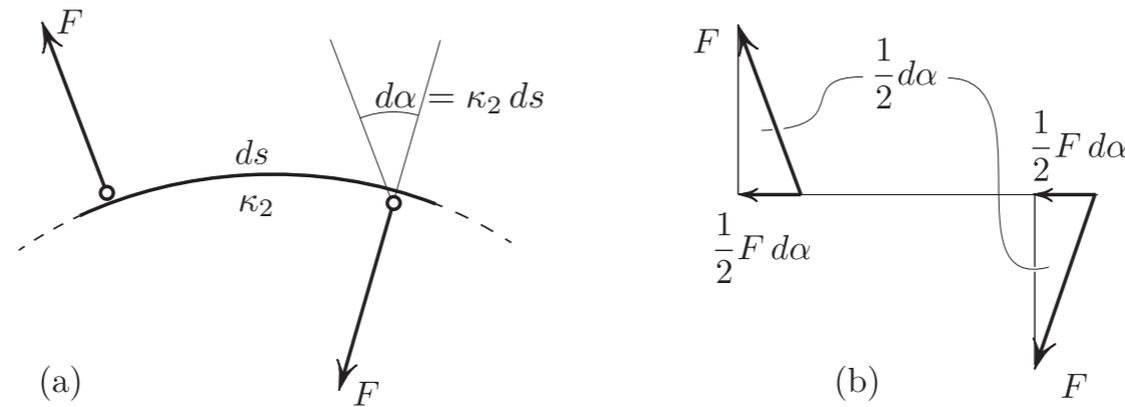
With the velocity  $\mathbf{v}$  pointing in the principal direction 1, the disk's plane rotates around the diameter tangent to line 2. The angular velocity of this rotation:  $\omega_{\text{diam}} = v\kappa_1$ , where  $\kappa_1$  is the principal curvature in the direction 1. Consequently, the disk, whose axial angular momentum is  $L$ , exerts the gyroscopic torque

$$\tau = L\omega_{\text{diam}} = Lv\kappa_1$$

against the tangency constraint. The torque's vector points out of the page, tangent to line 1.



## A heuristic explanation-continued:



The surface reacts with an equal and opposite torque, which can be considered to be due to the torque of two forces,  $\mathbf{F}_a$  and  $\mathbf{F}_b$ , applied by the surface to the disk. The torque of these forces points into the page.

The sum of these reaction forces points to the left in the figure, i.e. is normal to  $\mathbf{v}$ . This is precisely the “magnetic force” mentioned in the theorem.

To explain why  $F_{\text{deflection}} = L K v$  we first observe:

$$F_{\text{deflection}} = F d\alpha = F \kappa_2 ds,$$

where  $\kappa_2$  is the curvature of curve 2. But

$$F ds = \tau = Lv\kappa_1.$$

Substituting this into the previous expression gives

$$F_{\text{deflection}} = Lv\kappa_1\kappa_2 = LvK,$$

as claimed.

# Geodesics on vibrating manifolds and compound pendula

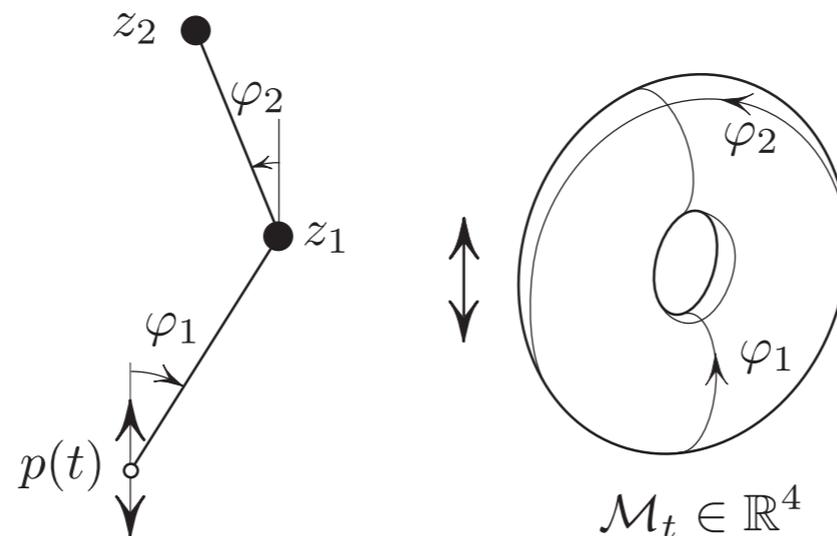
**Example:** A double pendulum.

Configuration space is  $\mathbb{T}^2$  parametrized by the angles  $(\varphi_1, \varphi_2)$ .

This  $\mathbb{T}^2$  is naturally embedded in  $\mathbb{R}^4 = \{z_1, z_2\}$ . The torus is “carved out” of  $\mathbb{R}^4$  by the two constraints on rod lengths:

$$|z_1 - p(t)| = L_1, \quad |z_2 - z_1| = L_2,$$

where  $p(t) \in \mathbb{R}^2$  is the position of the pivot.



If the pivot  $p(t)$  vibrates, so does our 2-torus  $\mathcal{M}_t \subset \mathbb{R}^4$  ( $\mathcal{M}$  for “manifold”).

# Geodesics on vibrating manifolds and compound pendula

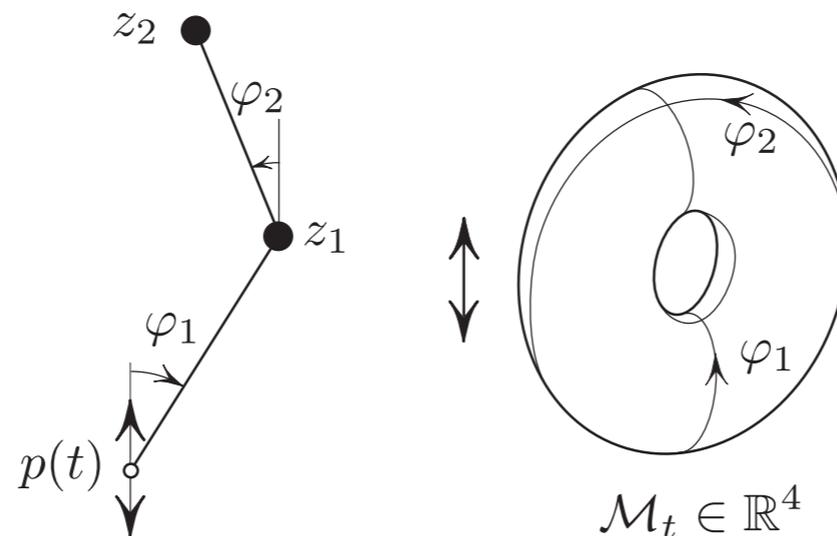
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If the pivot  $p(t)$  vibrates, so does our 2-torus  $\mathcal{M}_t \subset \mathbb{R}^4$  ( $\mathcal{M}$  for “manifold”).

**A general question of independent interest:** How does “vibration” of a manifold  $\mathcal{M}_t \subset \mathbb{R}^m$  affect geodesic motion?

## Geodesics on vibrating manifolds

Let  $\mathbb{R}^m$  be endowed with a metric  $\langle Mv, v \rangle$ .

Let  $\mathcal{M}_t$  be a one-parameter family of submanifolds of  $\mathbb{R}^m$ , of dimension  $n < m$ .

*How to coordinatize different  $\mathcal{M}_t$  in a consistent and meaningful (whatever it means) way?*

## Geodesics on vibrating manifolds

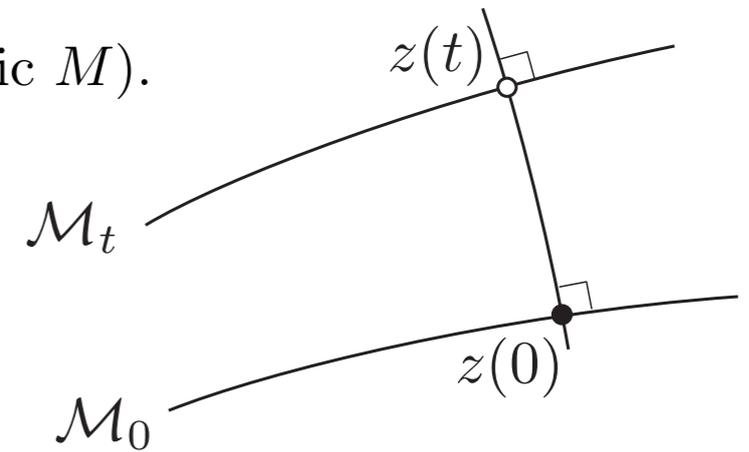
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*How to coordinatize different  $\mathcal{M}_t$  in a consistent and meaningful (whatever it means) way?*

**Definition.** A *normal curve*  $z(t)$  is defined by the conditions

- $z(t) \in \mathcal{M}_t$  for all  $t$
- $\dot{z}(t) \perp \mathcal{M}_t$  for all  $t$  ( $\perp$  is in the sense of the metric  $M$ ).



## Geodesics on vibrating manifolds

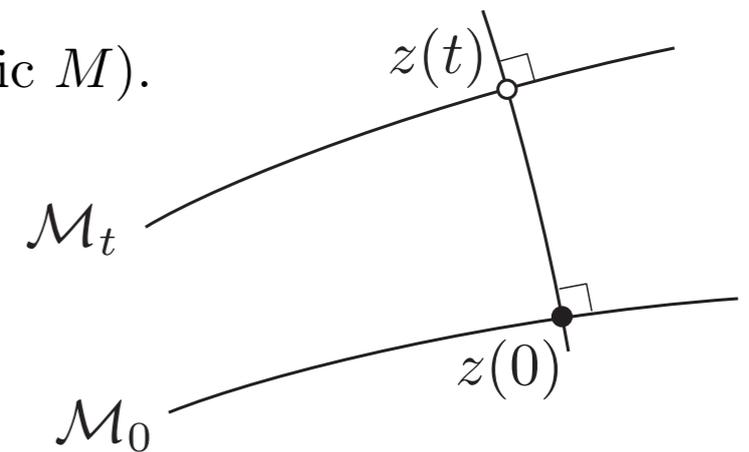
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**Definition.** A *normal curve*  $z(t)$  is defined by the conditions

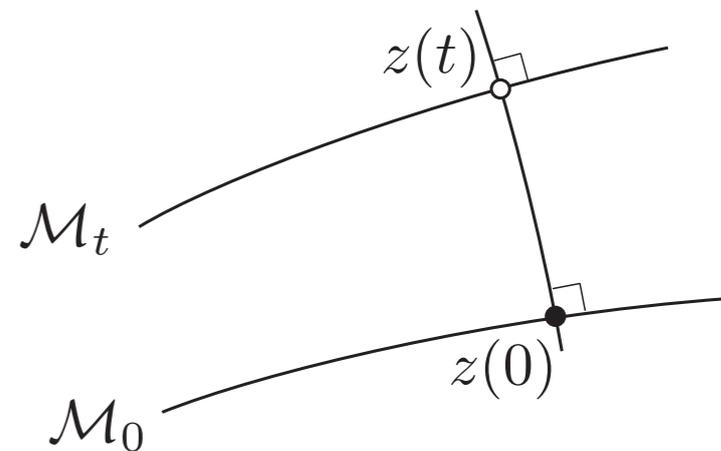
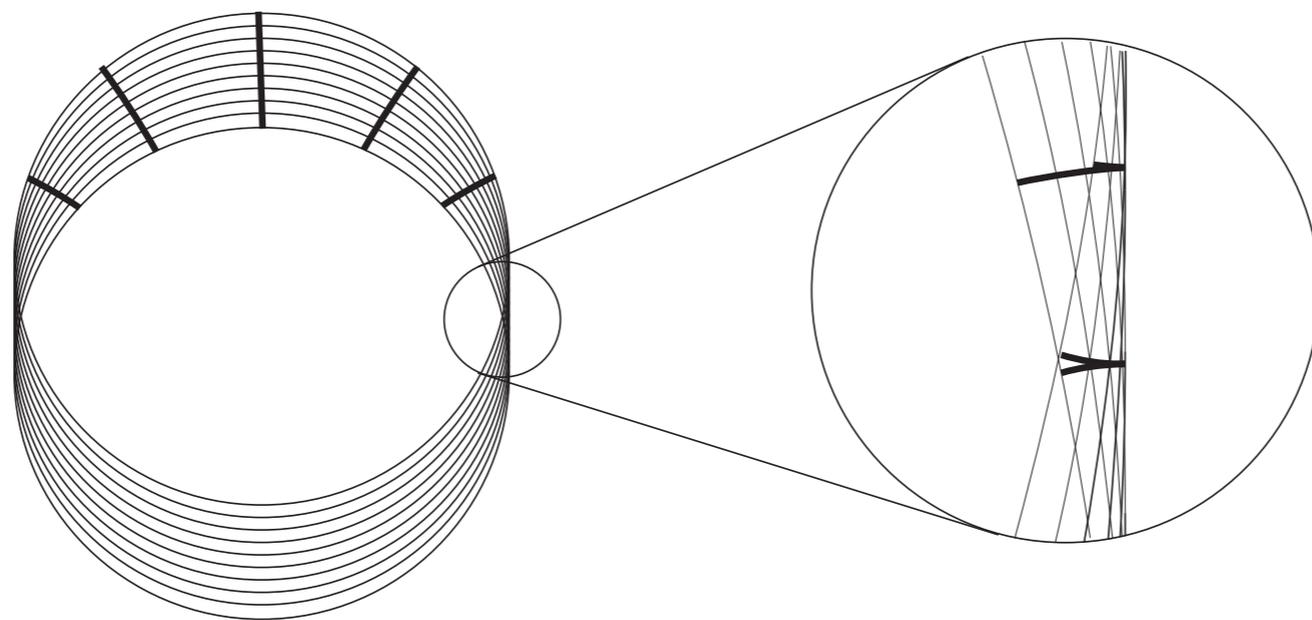
- $z(t) \in \mathcal{M}_t$  for all  $t$
- $\dot{z}(t) \perp \mathcal{M}_t$  for all  $t$  ( $\perp$  is in the sense of the metric  $M$ ).



What if codimension  $> 1$ ?

# Geodesics on vibrating manifolds

Examples of normal curves and impact theory.



## Geodesics on vibrating manifolds

Let  $\mathcal{M}_\tau = \mathcal{M}_{\tau+1}$ , and assume the “amplitude” of vibration is  $O(\varepsilon)$ .

**Theorem**(King-L) Consider a “geodesic” (i.e. a free particle)  $q(t)$  on  $\mathcal{M}_{t/\varepsilon}$  and the corresponding base point  $q_0(t)$  on  $\mathcal{M}_0$ . To the leading order in  $\varepsilon$ ,  $q_0$  behaves as a geodesic on  $\mathcal{M}_0$  with the additional force  $\overline{kv^2}N$ , where  $k$  is the curvature of the normal curve at  $q_0$ ,  $v$  is the normal velocity of the manifold and  $N$  is the principal normal vector to the normal curve.

Furthermore,

$$kv^2N = \nabla_T \frac{v^2}{2},$$

and thus the “shadow”  $q_0(t)$  behaves as an free particle subject to the potential force of the potential  $-v^2/2$ .

This shows that the fastest moving parts of the manifold are attracting.

Normal coordinates vs. the traditional choice.

Open question: stabilization of upside-down molasses.



Other items:

- A bike track connection
- Berry's phase
- Stokes phenomenon

**THANK YOU**