The Cauchy-Schwarz Inequality and a Paradox/Puzzle

MATHEMATICAL CURIOSITIES

By Mark Levi

The Inequality

Consider an asymmetric U-tube — two cylinders connected by a thin tube, as in Figure 1. As I depress the water in the right arm (using a piston, for example), I increase the water's potential energy.[1] Translated into algebra, this becomes the Cauchy-Schwarz inequality, as I will now demonstrate.

The potential energy of a cylindrical column of water of height h and radius r equals the weight times the height $h/2$ of the center of mass, i.e.,

$$kr^2h^2,$$

where $k=\pi\rho g/2$. Here, ρ is the water's density and g is the gravitational acceleration. From now on, we choose units in which $k=1$.

The new potential energy in Figure 1b is greater:

$$\Sigma r^2h^2 \geq \left(\Sigma r^2\right)\overline{h}^2, \quad (1)$$

where the subscripts $k=1,2$ are dropped and where $\overline{h}=\Sigma r^2h/\Sigma r^2$ is the average level in Figure 1a. Substituting this value of $\overline{h}$ into (1) gives

$$(\Sigma r^2h^2)(\Sigma r^2) \geq \left(\Sigma r^2h\right)^2. \quad (2)$$

This is the Cauchy-Schwarz inequality in disguise: setting $rh=x$, $r=y$ implies $r^2h=xy$ and turns (2) into

$$(\Sigma x^2)(\Sigma y^2) \geq (\Sigma xy)^2. \quad (3)$$

This works verbatim for sums of any number n of terms; one just needs to have n cylinders instead of just two [1].

[1] Potential energy increases precisely by the amount of work done in overcoming the hydrostatic pressure; friction is of course ignored.

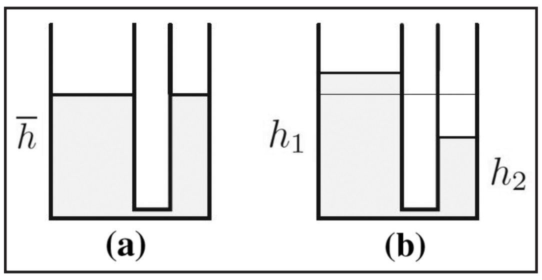

Figure 1. *Potential energy $P_{(b)} \geq P_{(a)}$; this is equivalent to (3).*

A Paradox

Consider a symmetric U-tube with water at rest, as in Figure 2. Using a piston, I push the right column of water down; the left column will rise by an equal amount. As much water goes down as up, and by the same distance. Therefore, *the average height of the water, i.e., the height of the center of mass, does not change, and neither does the potential energy*.

But this contradicts the earlier argument, as well as the following thought experiment. Instead of depressing the column, cut a cylinder of water off of the column's top and place this cylinder on top of the other column, thus achieving exactly the same configuration as when using a piston. Since the work done by lifting the cylinder is positive, so is the change in potential energy.

I offer the question in the caption of Figure 2 as a puzzle.

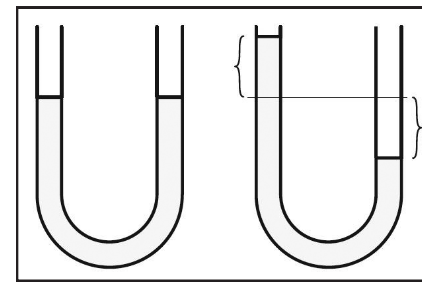

Figure 2. *When I pushed the water down in one arm of the tube, just as much water moved up the other arm, and by the same distance. In other words, the average vertical displacement is zero. The height of the center of mass thus remains unchanged. Where is the mistake in this short argument?*

The figures in this article were provided by the author.

References

[1] Levi, M. (2019). A water-based proof of the Cauchy-Schwarz inequality. *Am. Math. Monthly*, in press.

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