An Electrician's (or a Plumber's) Proof of Euler's Polyhedral Formula

MATHEMATICAL CURIOSITIES
By Mark Levi

Euler's famous polyhedral formula,

$$V - E + F = 2, \qquad (1)$$

describes the numbers of vertices, edges, and faces of a polyhedron "without holes," i.e., one that is sphere-like and in three dimensions. If the polyhedron has one hole, as in Figure 2, then we subtract 2 from the right-hand side of the formula, which becomes $V - E + F = 0$. The same thing happens for each additional hole (or, putting it differently, "handle on the sphere"). I will describe an argument based on electric circuits, leading to Euler's formula. I learned this beautiful idea from Peter Lax, and I therefore lay no claim of originality, except for any errors.

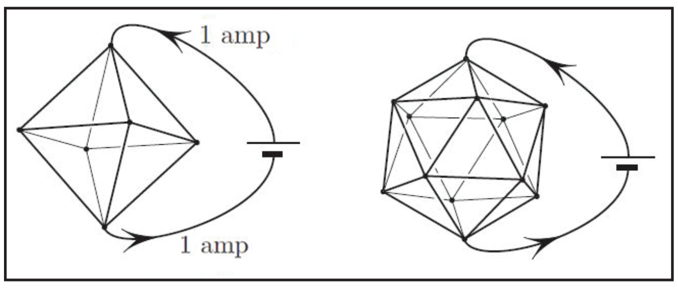

Figure 1. *V, E, and F are the numbers of vertices, edges, and faces of a polyhedron.*

Imagine our polyhedron as a wire frame, the edges being conducting wires, each of resistance 1 ohm, welded together at the vertices. Let us connect two vertices (chosen arbitrarily) to a battery, adjusting the voltage so as to drive the current of exactly 1 ampere. Now, Nature will pick a specific value for each edge's current. In doing so, she obeys Kirchhoff's laws: the currents satisfy some equations that determine the currents. Let us take it for granted that

the number of unknown currents =
the number of independent equations.

This sentence is already Euler's formula in disguise! Indeed, the left-hand side is E, one unknown current per wire. For the right-hand side, Kirchhoff's laws state the following:

(i) The sum of currents entering each vertex is zero, giving V equations. But one of these equations is redundant, since it results from adding up all the others (I leave out the simple verification), yielding $V-1$ equations.

(ii) The sum of voltage drops around each face is zero. This gives F equations, one of which is the sum of the remaining ones and thus redundant, for the total of $F-1$ equations.

Summarizing,

$$E = (V-1) + (F-1),$$

which amounts to (1). For a polyhedron with a hole, as illustrated in Figure 2, we must add two more equations, expressing the fact that the voltage drop over each of two non-contractible circuits is zero, resulting in $E = (V-1) + (F-1) + 2$, or

$$V - E + F = 0.$$

Admittedly the proof is not rigorous as given, since, for instance, I did not eliminate the possibility of more redundant equations.

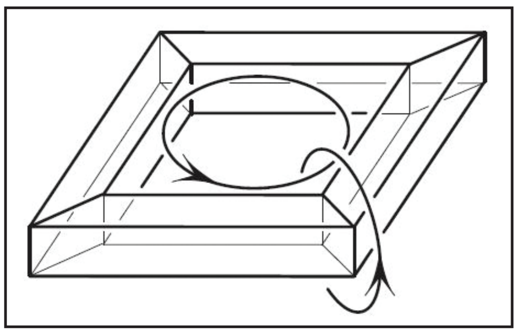

Figure 2. *A polyhedron with a hole; two non-contractible circuits are indicated.*

Although this proof would have sounded strange in Euler's pre-electricity days, it could be reformulated in plumber's terms by treating the polyhedron as a network of tubes (with porous blockages playing the role of resistors), currents as the mass flow per second, and voltages as pressures.

The figures in this article were provided by the author.

Mark Levi (levi@math.psu.edu) is a professor of mathematics at the Pennsylvania State University.